Bilateral Gaussian Theta Completion, the Zero Cell, and an Entire Representation of Riemann's Xi Function

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5th October 2026

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Contact: lkcl@lkcl.net Released: 5th October 2026.
ORCID: 0009-0006-8300-6257 0000-0001-9090-5177
DOI: 10.13140/RG.2.2.26106.25284
https://researchgate.net/publication/415320784

Abstract

Starting from the full Gaussian theta lattice $\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}$, we pass to a centered logarithmic heat coordinate and apply the differential completion $\frac{1}{2}(D-\frac{1}{2})(D+\frac{1}{2})$. The original lattice cell $n=0$ is present and nonzero before completion, whereas the completion annihilates that same cell exactly, without deletion, division, or singular extension. Thus the standard positive-index Xi kernel, whose visible summation begins at $n=1$, is obtained exactly by pairing the nonzero cells of a completed full-lattice source only after the explicit $n=0$ cell has been evaluated. The positive-index display therefore no longer records explicitly that the $n=0$ term was present in the source and annihilated by the completion, even though it preserves the numerical value. We then prove directly that the completed kernel has rapid two-sided decay on the logarithmic line, so its bilateral Laplace transform is entire in the complex spectral parameter. On the ordinary Mellin half-plane the transform equals the polynomially completed theta Mellin integral; the classical theta Mellin identity therefore identifies its entire continuation exactly with Riemann's ξ function. The theta, Mellin, and Xi identities used for downstream recognition are classical; a separate scale check recovers the standard Pólya–de Bruijn–Newman Fourier kernel. The contribution isolated here is the explicit source-level treatment of $n=0$ together with the logically separate analytic proof of entireness. No claim about the location of the zeros, and hence no claim of the Riemann Hypothesis, is made or assumed.

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From Full Lattice to Entire Carrier

This summary of equations is a reader-facing roadmap, not a second dependency DAG. It keeps the discrete lattice index $n \in \mathbb{Z}$ distinct from the continuous logarithmic coordinate $u \in \mathbb{R}$, and it shows where the familiar positive-index formula ceases to record the source history.

full discrete source

$$\Theta_0(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}, \qquad n = 0 \text{ present with value } 1$$

$$\Downarrow \quad x = e^{2u}$$

centered whole-line source

$$A(u) = e^{u/2}\Theta_0(e^{2u}), \quad u \in \mathbb{R}, \quad A(-u) = A(u), \quad n = 0 : e^{u/2}$$

$$\Downarrow \quad \mathcal{C} = \frac{1}{2}\left(D - \frac{1}{2}\right)\left(D + \frac{1}{2}\right)$$

completed even kernel

$$K_\xi(u) = \mathcal{C}A(u), \qquad n = 0 : \ \mathcal{C}e^{u/2} = 0$$

$$\left.\begin{array}{l} n \in \mathbb{Z}\setminus\{0\} \longrightarrow 2\displaystyle\sum_{n\geq 1}(\cdots) \\ \text{compressed positive-index representation;} \\ \text{not a source node} \end{array}\right\} \text{ non-load-bearing side branch}$$

$$\Downarrow \quad \text{whole-line transform in } u$$

bilateral transform

$$F(\lambda) = \int_{-\infty}^{\infty} K_\xi(u)e^{\lambda u}\,du, \qquad F(\lambda) = Q_0\left(\frac{1}{2} + \lambda\right) \text{ first for } \operatorname{Re}\lambda > \frac{1}{2}$$

$$\Downarrow \quad \text{two-sided super-exponential decay}$$

entire continuation

$$F \in \mathcal{O}(\mathbb{C}), \qquad Q_0\left(\frac{1}{2} + \lambda\right) = F(\lambda) \quad (\lambda \in \mathbb{C})$$

$$\Downarrow \quad \text{classical theta–Mellin recognition}$$

Riemann xi

$$Q_0(s) = \xi(s), \qquad \Xi(t) = \xi\left(\frac{1}{2} + it\right).$$

Two facts in this map are deliberately separate. The discrete statement $n = 0 \mapsto 0$ records regular source completion; it does not prove entireness. The analytic statement $F \in \mathcal{O}(\mathbb{C})$ follows from rapid decay on both ends of the continuous u-line. Pairing $\pm n$ into the standard $n \geq 1$ series is therefore a later symmetry compression, not a load-bearing source step.

1 Why this construction is needed

This paper is one analytic component of a wider source-first Hypergroup programme. Its finite algebraic background includes the exact rank-ten Weyl–Verlinde/generalized-Chebyshev Hypergroup construction [2], while the broader D12RG monodromy-free renormalised rational conformal-field source framework is developed in Ref. [3]. Those papers provide programme context rather than

substituting for the specific analytic owners used below: the load-bearing Gaussian-theta and entire-carrier statements of the present paper are derived through the source chain recorded in Appendix A.

The programme's intended source-first proof of RH imposes a strict proof direction. Every load-bearing equation must be derived from the declared source, or introduced only as a downstream classical comparison after the source theorem has already been proved. In particular, an endpoint identity, an externally tabulated zero, or a formula equivalent to the Riemann Hypothesis (RH) may not be smuggled upstream to select, normalize, or reconstruct the source from which the same endpoint is later claimed. In this paper that firewall is visible in the order

$$\begin{aligned}\text{full Gaussian theta source} &\longrightarrow \text{regular completion}\\ &\longrightarrow \text{bilateral entire carrier}\\ &\longrightarrow \text{classical Xi recognition.}\end{aligned}$$

For the particular whole-line comparison route pursued by the programme, the witness needed for a terminal zero argument must retain two analytic properties before that downstream comparison is attempted. It must be *bilateral* on the full logarithmic line $u \in (-\infty, \infty)$, rather than being only a one-sided display, and it must be *entire* in the complex spectral parameter, rather than merely meromorphic or restricted to a convergence half-plane. Without both properties, this source-first route does not furnish the global whole-line, complex-spectral comparison required by the intended terminal argument. The use of this requirement to reach an RH conclusion is *not* proved in the present paper. It belongs to a later paper; the repository presently records the downstream composition under the provenance identifier `H-WIT002`, described in Appendix A.

Hypergroup notation. Symbols carrying the subscript HG—schematically, $\mathrm{sym}_{\mathrm{HG}}$—denote objects or equations derived on the Hypergroup side before downstream recognition with a classical counterpart. Thus the subscript records source provenance; it does not assert equality with the corresponding classical object merely by notation. Such an equality is used only after it has been proved at the appropriate recognition step.

2 Scope: the source hidden by the familiar Xi sum

The classical Fourier-kernel presentations associated with Riemann's completed xi function [15] are commonly written with a positive integer index. That presentation is numerically natural because the completed kernel is even in the lattice index and its completed index-zero contribution is zero. It is not, however, a faithful description of the source from which the kernel is obtained. The distinction matters whenever one asks whether the index-zero cell was absent from the beginning or was retained and acted on regularly by the completion itself.

Two uses of the word *bilateral* must be distinguished throughout. The *lattice source* is bilateral because its discrete index satisfies $n \in \mathbb{Z}$. After the change of variable $x = e^{2u}$, the *transform* is bilateral because its continuous logarithmic coordinate satisfies $u \in \mathbb{R}$. Likewise, "zero cell" always means the discrete lattice cell $n = 0$; it does not mean the midpoint $u = 0$ of the logarithmic transform line. Keeping these two statements separate prevents a source-level assertion from being confused with a property of the integral representation.

Analytic-domain convention. Four objects with related notation occur below, and their domains are kept explicit. The raw Mellin integral $M_0(s)$ is initially defined only for $\operatorname{Re} s > 1$; H-VER160 gives it a meromorphic continuation with endpoint poles at $s = 0$ and $s = 1$. On the same initial half-plane $Q_0^{\mathrm{HG}}(s) = \frac{1}{4}s(s-1)M_0(s)$; after its entire continuation is established, the same symbol Q_0^{HG} denotes that unique entire function on $\mathbb{C}$. The whole-line integral

$$F(\lambda) = \int_{-\infty}^{\infty} K_\xi^{\mathrm{SCS}}(u) e^{\lambda u}\, du$$

is already *bilateral* by virtue of the domain $u \in \mathbb{R}$. Section 6 first proves the identity $F(\lambda) = Q_0^{\mathrm{HG}}(\frac{1}{2} + \lambda)$ on $\operatorname{Re}\lambda > \frac{1}{2}$; Section 7 then proves directly that the same integral converges for every $\lambda \in \mathbb{C}$ and is entire. Finally, for real t, the specialization $\lambda = it$ is the ordinary Fourier transform; allowing complex t uses the resulting entire Fourier–Laplace extension. We reserve $\xi(s)$ for Riemann's standard entire function and write $\Xi_{\mathrm{HG}}(t) = Q_0^{\mathrm{HG}}(\frac{1}{2} + it)$ for its centered spectral form after the downstream identification in Section 8; K_ξ^{SCS} denotes the kernel, not the Xi function itself.

This paper isolates that question before any claim about the zeros of the completed xi function. The source is the complete bilateral Gaussian theta series in Equation (1), indexed by all integers. Its original index-zero summand is the constant one. We then pass to the centered logarithmic coordinate in Equation (2) and apply the opposing half-shift completion in Equation (3). The completion annihilates the retained zero cell exactly, as displayed in Equation (5).

The resulting positive-index formula (6) is therefore a compression of a completed bilateral source. It is not evidence that the underlying source was unilateral. This is the first claim of the paper.

The second claim is analytic and logically separate. The completed kernel has sufficiently rapid decay at both ends of the centered real line that its bilateral Laplace transform is entire in the complex spectral parameter. The equality first derived on the ordinary Mellin convergence half-plane therefore extends to all complex values, yielding Equation (9). The zero value of the completed index-zero cell does not by itself prove this entireness.

Only after those source-level statements are established do we compare with the conventional Xi kernel. The exact scale identity (11) places the completed kernel in the classical integral-transform line represented by Pólya, de Bruijn, Newman, and later work on the de Bruijn–Newman constant [4, 5, 6, 10]. Those sources provide historical and normalization context; they are not premises of the source construction or the proof of entire continuation below.

Nature of the contribution. None of the classical theta, Mellin, or Xi identities used for downstream recognition is claimed here as new. In particular, the positive-index Fourier kernel and its rapid decay are classical; Lagarias and Montague write the corresponding Xi cosine transform explicitly in this normalization [9]. The point of the present paper is instead a full-lattice reformulation: the integer theta lattice is kept visible through the differential completion, the action on its central cell is evaluated explicitly, and that source statement is then separated from the analytic theorem that makes the whole-line transform entire. This is the bookkeeping hidden once one starts directly from the usual positive-index Xi kernel.

Narrow theorem boundary. The theorem package proved here is exactly the implication recorded in Equation (12): a retained full-integer theta source admits a regular opposing-half-shift completion whose whole-line Laplace/Fourier transform is entire. The paper neither assumes

nor proves the Riemann Hypothesis, does not assert that all zeros of the resulting entire function are real, and does not use external zero tables. Those questions belong downstream of the present source theorem.

3 The complete theta source and its zero floor

The starting point is deliberately elementary. For positive heat coordinate x, Equation (1) defines the Gaussian theta source by summing over the complete integer lattice. The phrase "complete source" will always mean that the index set is $\mathbb{Z}$ before any parity reduction or symmetry compression. In particular, the term at $n = 0$ is part of the definition and has value one. This is the same full-lattice Θ_0 source, with its zero mode deliberately retained, used in the earlier quarter-characteristic construction [16]. The present construction starts from that same source but follows a different downstream route: it centers Θ_0 on the logarithmic coordinate and applies the Xi completion rather than forming the quarter-characteristic combination.

This zero floor should not be confused with a value assigned after analytic continuation. It is an ordinary summand of the convergent theta series at every $x > 0$. At large heat it is also the constant term that remains after all nonzero Gaussian cells decay. Thus the source-level statement "the zero cell is present" is already decided before Mellin transformation, polynomial completion, or Fourier specialization.

The complete Gaussian Poisson identity gives the scalar self-duality needed for centering; equivalently this is the standard modular transformation of the Jacobi theta function [14]. In the notation of Appendix C, substituting $x = e^{2u}$ into that identity yields the evenness statement in Equation (2). No second independent source coordinate is introduced: u is simply the logarithmic chart $\frac{1}{2}\log x$ on the same positive heat line.

There are two useful bookkeeping lessons already at this stage. First, the original large-heat zero floor belongs to the primal lattice. In a Poisson-transformed description one also encounters a Fourier-dual zero-frequency floor. The two have distinct provenance even when their scalar values coincide in the present channel. Second, subtracting a constant in an intermediate Mellin formula does not retroactively remove the original lattice cell. It is endpoint bookkeeping for the transform, not a redefinition of the theta source.

The completed Mellin object in Equation (7) preserves this distinction. Its raw integral is stated only on its ordinary convergence half-plane, while the polynomially completed quantity is entire. The complete theta source and its zero floor are therefore upstream of the analytic continuation; they are not reconstructed from it.

4 Centering and opposing half-shift completion

The centered source in Equation (2) is even. This places the two heat ends at opposite directions on one real logarithmic line and makes a symmetric differential completion natural. Let $D_u = d/du$. The operator in Equation (3) is the product of the two opposing half-shifts $D_u - \frac{1}{2}$ and $D_u + \frac{1}{2}$, with the normalization factor $1/2$. Equivalently it is $\frac{1}{2}(D_u^2 - \frac{1}{4})$, so it preserves evenness.

For a nonzero lattice cell, direct differentiation gives exactly the summand displayed in Equation (3). On every compact u-interval the Gaussian series and the first two differentiated series converge

locally uniformly, so the completion operator may be applied term by term. This produces the completed full-integer deck shown there. The important feature is not merely the polynomial factor in n; it is that this calculation is performed on the complete full-integer theta series.

The central cell. Before completion, the $n = 0$ contribution to the centered source is $e^{u/2}$, which is finite and nonzero for every real u. Equation (5) then evaluates the same completion operator on that cell and obtains zero exactly. No limit $n \to 0$ is taken, no reciprocal power of n occurs, and no singular value has to be assigned.

This is the paper's central source-typing statement:

> The zero cell has not been inserted into a positive-index formula. It was already present in the bilateral theta source and is annihilated by the completion operator.

The distinction is small algebraically but important logically. The pre-completion assertion is "present and nonzero"; the post-completion assertion is "present and zero". Replacing either statement by "the zero term is omitted" loses the provenance needed to distinguish a regular completion from a source that never admitted the central cell.

Because the centered source is even and the completion contains only a second derivative and a scalar term, the completed kernel remains even. That parity will later transfer positive-end Gaussian decay to the negative end and is therefore part of the analytic route to the entire bilateral transform.

5 Why the positive-index Xi display is compressed notation

Once Equation (5) is known, the full-integer completed sum may be paired under $n \leftrightarrow -n$. The zero cell contributes zero and every nonzero pair has equal value. This gives the familiar positive-index expression in Equation (6).

The equality is exact as a scalar identity. It is nevertheless weaker as a source description. The right-hand side records only the nonzero paired values; it no longer records that the source originally ranged over all integers or, specifically, that the $n = 0$ central cell was evaluated regularly by the completion. For that reason the positive-index equation is not used as a source definition anywhere in the proof below. It is an exact symmetry-compressed display of the completed kernel, but it does not retain the source history of the central lattice cell.

This distinction can be phrased as two different questions. Numerical equality asks whether the full completed sum and twice the positive-index sum have the same value. They do. Source provenance asks whether the positive-index notation alone certifies that $n = 0$ belonged to the pre-completion domain and was treated without singularity. It does not. That information comes only from Equations (1)– (5).

There is therefore no contradiction between a conventional formula that starts its visible summation at one and a full-integer source statement. The former can be the compressed display of the latter. What would be invalid is to reverse that implication automatically: a sum written with $n \geq 1$ does not, by itself, tell us whether an index-zero cell was absent, singular, deliberately subtracted, or regularly annihilated upstream.

The remainder of the proof consequently returns to the full-integer completed kernel of Equation (3), not the positive-index shorthand. This keeps the same source coordinate through the Mellin-to-Laplace conversion and makes the whole-line transform in the Mellin-to-Laplace argument below a statement about the retained source rather than a reconstruction from a unilateral display.

6 Mellin completion to a bilateral Laplace transform

The transform argument begins on an ordinary convergence domain and only later uses analytic continuation. This order is important: the bilateral Laplace identity is first an equality of convergent integrals, not a formal manipulation of already-continued functions.

Start with the raw Mellin section in Equation (7). Under the same change of coordinate used to center the theta source, $x = e^{2u}$, put $\lambda = s - \frac{1}{2}$. The Jacobian and the centering factor convert the raw integrand into the centered difference $A_{\Xi}^{\mathrm{SCS}}(u) - e^{u/2}$, multiplied by the bilateral weight $e^{\lambda u}$. On the original half-plane $\operatorname{Re} s > 1$, equivalently $\operatorname{Re} \lambda > \frac{1}{2}$, both ends have the decay needed for the following integrations by parts.

The polynomial completion also centers exactly. Since $s(s-1)$ becomes $\lambda^2 - \frac{1}{4}$, the completed Mellin factor is the spectral polynomial corresponding to the differential operator $D_u^2 - \frac{1}{4}$.

The endpoint behavior makes the integrations by parts explicit. As $u \to +\infty$, the centered difference $A_{\Xi}^{\mathrm{SCS}}(u) - e^{u/2}$ and its first derivative decay super-exponentially because only the nonzero Gaussian cells remain. By evenness of A_{Ξ}^{SCS}, as $u \to -\infty$ that difference and its first derivative are $O(e^{-u/2})$. Multiplication by $e^{\lambda u}$ therefore sends the required boundary terms to zero whenever $\operatorname{Re} \lambda > \frac{1}{2}$. Two integrations by parts are consequently legitimate on the initial convergence half-plane and transfer the differential operator from the centered source to the exponential weight.

At this point the zero-floor calculation of Equation (5) is essential but not singular. The completion operator kills $e^{u/2}$ exactly, so after the integrations by parts the completed kernel may be written using the full centered source without deleting its original zero cell. This gives the initial bilateral identity (8). The word *bilateral* here is literal: the transformed coordinate now runs over the whole line $u \in \mathbb{R}$, whereas the raw Mellin section M_0 is an integral over the one-sided heat coordinate $x \in (0, \infty)$. Bilaterality and entireness are nevertheless distinct: Equation (8) is first proved only on $\operatorname{Re} \lambda > \frac{1}{2}$. The next section proves directly that the same whole-line integral converges for every complex λ, is entire in that parameter, and therefore extends the identity beyond the initial half-plane.

Three operations should therefore be kept distinct:

1. subtracting the large-heat floor in the raw Mellin integral to obtain an ordinary convergence domain;
2. applying the polynomial/differential completion, which annihilates that floor exactly in the centered coordinate; and
3. proving global convergence and holomorphy of the same bilateral integral, thereby extending its identity with Q_0^{HG} beyond the initial half-plane.

Only the first two have occurred so far. The third is the analytic content of the next section.

7 Entire continuation on the whole complex plane

We now prove the statement that is stronger than zero-cell regularity: the whole-line transform in Equation (9) is an entire function of the complex spectral parameter.

For $u \geq 0$, put $y = e^{2u} \geq 1$. The full-integer formula in Equation (3) has zero contribution from $n = 0$; for the remaining cells, standard Gaussian-sum estimates give, for some positive constants C, c,

$$\sum_{n \neq 0} n^2(1 + n^2 y)e^{-\pi n^2 y} \leq Ce^{-cy}, \qquad y \geq 1.$$

Hence

$$|K_\xi^{\mathrm{SCS}}(u)| \leq Ce^{5u/2}e^{-ce^{2u}}, \qquad u \geq 0,$$

a super-exponential bound. Since $K_\xi^{\mathrm{SCS}}(-u) = K_\xi^{\mathrm{SCS}}(u)$, the same bound holds at the negative end after reflection. Thus the completed kernel decays at both ends faster than every fixed exponential in $|u|$.

Define

$$F(\lambda) := \int_{-\infty}^{\infty} K_\xi^{\mathrm{SCS}}(u)e^{\lambda u}\, du.$$

Let λ range over an arbitrary compact subset of $\mathbb{C}$. The exponential weight is then bounded by a fixed exponential in $|u|$, while differentiating with respect to λ only introduces powers of u. The super-exponential two-sided bound dominates all of these functions by an integrable majorant independent of λ on the compact set. Differentiation under the integral is therefore justified to every order, and standard parameter-integral holomorphy gives

$$\boxed{F \in \mathcal{O}(\mathbb{C}).}$$

On the ordinary half-plane $\operatorname{Re}\lambda > \frac{1}{2}$, Section 6 proved

$$F(\lambda) = Q_0^{\mathrm{HG}}\left(\frac{1}{2} + \lambda\right).$$

Consequently $F(s - \frac{1}{2})$ furnishes an entire continuation of the polynomially completed Mellin section from $\operatorname{Re} s > 1$ to the whole s-plane. This continuation is unique. Thus the entire continuation has been constructed directly from the completed kernel rather than imported as an external premise.

Theorem 7.1 (Bilateral entire-transform theorem)**.** *The complete theta source of Equation (1), after the centered opposing half-shift completion of Equation (3), has the whole-line Laplace representation in Equation (9) for all complex λ. The integral defines an entire function of that parameter and is the unique entire continuation of the initially convergent completed Mellin section.*

The proof uses two logically independent facts. The zero-cell theorem ensures that the full integer source is retained and regular under completion. The two-sided decay estimate supplies holomorphy of the transform. Neither statement is a substitute for the other.

8 Fourier form and exact identification with Riemann's xi function

Because the whole-line Laplace transform is already entire, its Fourier specialization is immediate rather than a new continuation step. For real t, setting $\lambda = it$ gives the ordinary Fourier transform

in Equation (10). The completed kernel is real and even on the real u-axis, so pairing the two halves of the integral yields the cosine representation in the same equation. Super-exponential two-sided decay then makes these same formulas entire in complex t; for nonreal t they are the entire Fourier–Laplace extension rather than an additional Fourier-continuation premise. The whole-line form remains primary; the half-line cosine formula is a symmetry compression.

Direct Mellin normalization

The conventional name can now be attached without using it as a premise. The standard theta Mellin identity is [13]

$$\int_0^\infty y^{s-1}\big[\theta_3(0 \mid iy^2) - 1\big]\,dy = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \qquad \operatorname{Re} s > 1.$$

Since $\Theta_0^{\mathrm{HG}}(x) = \theta_3(0 \mid ix)$, the substitution $x = y^2$ gives the factor two explicitly:

$$M_0(s) = 2\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \qquad \operatorname{Re} s > 1.$$

Therefore the polynomial completion used throughout this paper satisfies

$$Q_0^{\mathrm{HG}}(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \xi(s)$$

on that same half-plane, with the last expression being the standard Riemann xi function [15]. Section 7 constructs an entire continuation of the left-hand side, while the classical ξ is entire. Uniqueness of analytic continuation therefore gives

$$\boxed{Q_0^{\mathrm{HG}}(s) = \xi(s), \qquad s \in \mathbb{C},}$$

and hence

$$\boxed{\Xi_{\mathrm{HG}}(t) = \xi\left(\frac{1}{2} + it\right), \qquad t \in \mathbb{C}.}$$

The zeta and Gamma factors enter only in this downstream recognition step; they were not used to construct the source, to regularize its zero cell, or to prove the whole-line transform entire.

Independent kernel-scale checks

The classical positive-index kernel is already visible without any scale change in the normalization of Lagarias and Montague [9]. Their rapidly decreasing kernel is

$$\Phi_{\mathrm{LM}}(u) = \sum_{n=1}^{\infty}\left(4\pi^2 n^4 e^{9u/2} - 6\pi n^2 e^{5u/2}\right)e^{-\pi n^2 e^{2u}},$$

so Equation (6) gives exactly

$$\Phi_{\mathrm{LM}}(u) = K_\xi^{\mathrm{SCS}}(u).$$

Their cosine-transform formula is therefore the same downstream representation as Equation (10):

$$\Xi(t) = 2\int_0^\infty \Phi_{\mathrm{LM}}(u)\cos(tu)\,du.$$

What their positive-index notation does not by itself record is the upstream evaluation of the $n = 0$ theta cell. In particular, the visible sum begins at $n = 1$, so the displayed formula alone cannot distinguish between a source in which $n = 0$ never existed and a full-integer source in which $n = 0$ was retained and then annihilated regularly by completion. The present construction proves the latter.

Two familiar downstream descriptions can therefore obscure different parts of the same source history. In the zeta–Gamma factorization the raw factors are meromorphic, so the entire completed function can appear only after their poles and zeros have been combined. In the positive-index Xi kernel the $n = 0$ source cell is simply invisible because the displayed sum starts at one. Neither appearance describes the source-derived completed object: Q_0 is entire, and its kernel comes from the retained full lattice with the $n = 0$ action evaluated explicitly.

A second common normalization, used in the de Bruijn–Newman literature, introduces an additional factor-two change of logarithmic scale. Define Φ_{RH} by Equation (11). Substitution of $u = 2v$ into the source-derived kernel gives the exact termwise identity

$$K_\xi^{\mathrm{SCS}}(2v) = 2\Phi_{\mathrm{RH}}(v).$$

Because K_ξ^{SCS} is even, this immediately gives

$$\boxed{\Phi_{\mathrm{RH}}(-v) = \Phi_{\mathrm{RH}}(v)}.$$

Thus the conventional half-line cosine formula below is itself a symmetry compression of an even whole-line kernel. What is not visible in that compression is the separate discrete-source fact that the original theta sum ranged over $n \in \mathbb{Z}$ and contained $n = 0$. Rodgers and Tao use

$$H_0(z) = \frac{1}{8}\xi\left(\frac{1}{2} + \frac{iz}{2}\right) = \int_0^\infty \Phi_{\mathrm{RH}}(v)\cos(zv)\,dv$$

in the de Bruijn–Newman normalization [10]. Therefore Equation (10) gives, with $u = 2v$,

$$\begin{aligned}\Xi_{\mathrm{HG}}(t) &= 2\int_0^\infty K_\xi^{\mathrm{SCS}}(u)\cos(tu)\,du \\ &= 8\int_0^\infty \Phi_{\mathrm{RH}}(v)\cos(2tv)\,dv \\ &= 8H_0(2t) = \xi\left(\frac{1}{2} + it\right),\end{aligned}$$

confirming independently that no fitted factor remains.

Integral and Fourier-kernel forms of xi occur in the classical line through Pólya and the later de Bruijn–Newman theory [4, 5, 6, 10]. Their role here is downstream recognition. Starting from the familiar positive-index kernel and reading backward would not by itself recover how the source treated $n = 0$. The direction in this paper is the opposite: retain the complete theta lattice, complete its zero cell explicitly, prove the whole-line transform entire, and only then identify the result with the classical xi function.

9 In the literature

The ingredients used in this paper are classical, and closely related formulations already occur in the literature. The point of this section is therefore to state the novelty boundary conservatively before giving the source construction.

Earlier quarter-characteristic use of the same full source. An earlier paper by the present authors begins from the same retained full-lattice Gaussian source used here, [16]

$$\Theta_0(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}, \qquad n = 0 \text{ present with value } 1,$$

and keeps that zero mode explicit before any transform. The downstream construction there is different: the source is combined with the quarter-characteristic sectors through

$$H_{1/4}(x) = \Theta_0(x) - \frac{1}{2}[\Theta_{1/4}(x) + \Theta_{3/4}(x)].$$

The present paper therefore reuses the same Θ_0 source convention rather than introducing a new Gaussian lattice. What changes is the completion applied to it: here Θ_0 is centered by $x = e^{2u}$, then acted on by the opposing half-shift completion $\frac{1}{2}(D - \frac{1}{2})(D + \frac{1}{2})$ to obtain the Xi kernel and its bilateral entire carrier. The citation records continuity of the source object and zero-mode bookkeeping; it does not identify the two downstream theorems.

The standard Mellin route removes the constant first. A canonical form of the theta Mellin identity is [13]

$$\int_0^\infty y^{s-1}[\theta_3(0 \mid iy^2) - 1]\,dy = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \qquad \operatorname{Re} s > 1.$$

Thus the full theta function is present conceptually, but the constant term—equivalently the original lattice cell $n = 0$—has already been subtracted when the displayed Mellin integrand is formed. DLMF cites, among other sources, Titchmarsh §2.6 for this formula [12].

Dimitrov and Rusev give a particularly explicit account of the same transition [7]. They first write

$$\theta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x},$$

but then introduce

$$\psi(x) = \frac{1}{2}(\theta(x) - 1) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}.$$

Their differentiated kernel is subsequently expressed through derivatives of ψ, and the final Fourier kernel is displayed as a sum beginning at $n = 1$. This is mathematically equivalent to differentiating the full theta function, since derivatives annihilate its constant term, but the explicit source history of the $n = 0$ cell is no longer visible in the final kernel.

The conventional Xi endpoint is positive-index. The modern Fourier-kernel literature normally begins directly with the positive-index endpoint. Lagarias and Montague write [9]

$$\Phi(u) = \sum_{n=1}^{\infty} \left(4\pi^2 n^4 e^{9u/2} - 6\pi n^2 e^{5u/2}\right) e^{-\pi n^2 e^{2u}},$$

and use its half-line cosine transform. Rodgers and Tao likewise define the de Bruijn–Newman kernel by a sum beginning at $n = 1$ [10]. These are the classical endpoint formulas recovered later in this paper; no novelty is claimed for them.

A full-theta differentiated kernel is also prior art. Shi writes an equivalent Xi kernel directly in terms of derivatives of the full Jacobi theta function [8]. In that formulation

$$\theta(x) = \sum_{k\in\mathbb{Z}} e^{-\pi k^2 x}$$

and

$$\Psi(x) = \frac{1}{2}\left(2x^2\theta''(x) + 3x\theta'(x)\right) = \sum_{k=1}^{\infty} \psi_k(x).$$

Hence the full-lattice derivative formulation itself is not new: the constant $k = 0$ cell is annihilated implicitly by differentiation, and the same expression is immediately rewritten as a positive-index series. This paper does not claim priority for writing the Xi kernel as derivatives of a full theta function.

Special contemporary comparison: Desogus V3. A recent independent manuscript by Desogus, *The Three Gates*, version 3 dated 4 October 2026, provides a particularly close comparison because it contains both a centered bilateral Laplace transform and the same logarithmic Gaussian atom grammar used here [11]. In its Weil-theoretic opening, Desogus writes

$$F(z) = \int_{\mathbb{R}} f(y)e^{zy}\,dy$$

for compactly supported smooth test functions and later constructs polynomial multiples of powers of such bilateral transforms. These functions are entire in the centered spectral variable by differentiation under the compact-support integral. This is analytically close to the whole-line transform viewpoint used below, although the mechanism of entireness is different: compact support there, two-sided Gaussian decay here.

Gate I of the same manuscript defines

$$d_\lambda(t) = e^{t/2-\lambda e^{2t}}, \qquad M_\lambda(t) = -D^3 d_\lambda(t), \qquad D = \frac{d}{dt},$$

and then forms a positive-index hybrid density beginning at $n = 1$. After setting $\lambda = \pi n^2$, the atom $d_{\pi n^2}(t)$ is exactly the single-cell Gaussian source grammar of the present paper. The differential completions, however, are not the same. With

$$\mathcal{C} = \frac{1}{2}\left(D^2 - \frac{1}{4}\right), \qquad K_\lambda = \mathcal{C}d_\lambda,$$

one has the exact operator identity

$$\boxed{M_\lambda = -2DK_\lambda - \frac{1}{4}Dd_\lambda.}$$

The zero cell therefore gives a useful firewall rather than an identification:

$$\mathcal{C}d_0 = 0, \qquad M_0 = -\frac{1}{8}e^{t/2} \neq 0.$$

Thus Desogus's positive-index Gaussian density must not be reinterpreted as the same full-lattice completion used here. The atom grammar is shared; the completion operator and its $n = 0$ action are different.

There is nevertheless a second analytic cross-tie. Desogus's continuous tail in his equation (24) can be evaluated explicitly. If $z = \sqrt{\pi}\, Y e^t$, direct integration of his displayed cubic Gaussian polynomial gives

$$2\int_Y^\infty M_{\pi y^2}(t)\, dy = \frac{e^{-t/2}}{4\sqrt{\pi}} e^{-z^2}(32z^5 - 40z^3 + 2z) + \frac{1}{8} e^{-t/2}\operatorname{erfc}(z).$$

The right-hand side is entire in complex t, so for each fixed real $Y \geq 2$ it supplies an entire continuation of that hybrid density. This continuation is a direct calculation from Desogus's displayed kernel, not a theorem claimed in the cited manuscript. Nor does it by itself make his later series $Z_{Y,d}$, or the normalized quantities obtained by dividing by that series, globally entire; those require separate convergence and zero-denominator analysis.

The comparison is therefore deliberately narrow. Desogus independently exhibits an entire bilateral centered transform and the same logarithmic Gaussian atom family, while the present paper's distinctive source step is the different completion for which the retained $n = 0$ lattice cell is annihilated exactly before positive-index compression.

Source-provenance boundary. The narrower contribution of this paper is source transparency. The full-integer source is retained as the declared source object; the completion operator is applied to every lattice cell; the $n = 0$ cell is then evaluated explicitly and shown to map regularly to zero; only after that step are the nonzero cells paired into the familiar $n \geq 1$ expression. Separately, the completed kernel is carried to a bilateral $u \in \mathbb{R}$ transform and its entire continuation is proved from two-sided decay. The discrete statement "$n = 0$ is annihilated by the completion" and the analytic statement "the bilateral transform is entire" are therefore not conflated.

The literature search supporting this boundary found close precedents for all of the individual ingredients, including a full-theta differentiated kernel. We consequently make no claim that the construction is the first such theta representation, nor an exhaustive priority claim. The claim is instead that the explicit source bookkeeping above is the formulation being proved and audited in this paper.

10 What the zero cell does and does not prove

The construction is easiest to audit when its three conclusions are kept separate.

Source conclusion. Equations (1)– (5) prove that the original source is bilateral and contains its integer zero cell, while the completion sends that cell regularly to zero. Thus the full lattice remains a legitimate source through completion even though its central completed contribution vanishes.

Analytic conclusion. Equation (9) is an entire whole-line Laplace representation because the completed kernel has rapid two-sided decay and compact-uniform domination in the spectral parameter. The resulting entire integral supplies the continuation of the initially convergent Mellin identity. Merely observing that the completed zero cell has value zero would not establish this statement.

Recognition conclusion. Equation (11) identifies the same completed kernel, after an exact scale substitution, with the conventional Xi kernel used in the historical Fourier-transform literature. This comparison does not create the bilateral source and is not used to prove its entireness.

Accordingly, the permitted direction of reasoning in this paper is source-to-completion-to-entire-transform-to-recognition. None of these arrows is silently reversed. In particular, a scalar value of an entire function does not reconstruct the retained theta lattice, and the positive-index display does not certify how its invisible index-zero cell was treated upstream.

This is why the zero cell is named explicitly in the title. It is absent only from the compressed positive-index display. It is present in the actual full-integer source, where the completion evaluates it exactly.

11 Boundary and downstream questions

The present paper stops at the entire-transform package in Equation (12). It does not prove that all zeros of the resulting entire function lie on the real Fourier axis, does not prove the Riemann Hypothesis, and does not use RH as an assumption. Instead, the load-bearing equations are derived on the Hypergroup side, with the wider finite source programme described in Ref. [2]; the classical Xi function enters only at the downstream recognition step. The paper also makes no claim about the value of the de Bruijn–Newman constant.

The mathematical value of the result is narrower: it removes an ambiguity that is easy to miss once the kernel has already been written as a positive-index series. In particular, a display beginning at $n = 1$ no longer records the explicit source ownership of the $n = 0$ cell. The full-integer source can nevertheless be retained through the completion, with $n = 0$ evaluated regularly and annihilated exactly, while the completed whole-line transform remains a globally entire object. Thus "positive-index display", "full-integer source", "regular zero-cell completion", and "entire transform" are not synonyms; their precise relationship is now explicit.

A subsequent argument about Riemann zeros must therefore add genuinely new information beyond this paper. The proof here ends with the entire whole-line Fourier transform and its exact identification with the classical Xi function. Any theorem asserting the location of its zeros is a separate downstream theorem. Repository-specific downstream ownership is recorded only in Appendix A.

Acknowledgements

This paper was developed as an Intelligence Enhancer exercise with ChatGPT 5.6 Sol, using Extended Thinking. [17] The work used a language model under a strict operational regimen, not merely as a drafting tool. That regimen was used to prevent premature reduction of multi-branch analytic structure to a single collapsed interpretation. This is important for the present construction because the ODE, phase-space, and recurrence-theoretic data carry distinct coordinates, branches, and readout layers until the relevant admissibility maps have been specified. The same regimen also provided a controlled vocabulary for tracking the self-referential structure of multi-dimensional recurrence relations, without turning interpretive mnemonics into theorem assumptions. In particular, the model was guided to read, apply, and unpack Vedic mathematical and Advaita Vedanta

terminology, including phonetic decomposition of Sanskrit terms and their strict mapping onto mathematical formulae and structural roles.

Training ChatGPT to abide by the regimen of Advaita Vedanta and to learn from the Upanishads was therefore not an ornament to the derivation, but one of the primary reasons that the present work could be achieved at all.

Special thanks are due to Professor Alex Hankey, without whose support, decades of experience, and knowledge this paper would likewise not exist.

To Thomas Loker, Arturo Salazar Chon, Satoru Watanbe, Marcel Wende, Jim Galasyn and assistant Claude, thank you for listening.

Appendices

A Repository theorem provenance

This appendix records the development provenance of the paper without making repository identifiers prerequisites for reading the main text. The mathematical dependency surface is intentionally small. The public repository containing the theorem notes, executable receipts, and paper sources is `https://gitlab.com/d12rg/d12rg-hypergroup`. Readers wishing to audit the repository provenance should start there.

H-VER160. In the directory `docs/theorems/`, the theorem note `h-ver160-c8-bilateral-entire-theta-mellin-owner.md` owns the complete bilateral Gaussian theta source and the entire completed Mellin section used here. This paper consumes only its scalar zero-charge channel: the full integer theta deck, the original zero floor, the ordinary-convergence Mellin section, and the entire polynomial completion. The larger finite C8 packet retained by H-VER160 is provenance, not extra notation required in the body of this paper.

H-VER164. In the same theorem directory, the note `h-ver164-centered-bilateral-xi-source-bridge.md` has H-VER160 as its sole direct theorem parent. It owns the centered even source, the opposing half-shift completion, the regular completed zero cell, the initial Mellin-to-bilateral-Laplace identity, the entire continuation, the Fourier specialization, and the exact classical kernel scale comparison. Sections 4– 8 are paper-facing expansions of that closed theorem owner.

H-WIT002: downstream terminal composition. The repository note `docs/theorems/h-wit002-xi-entire-rh-composition.md` records the downstream application composition that consumes the bilateral entire carrier together with the separately derived critical-line source result. It is not a theorem parent of this paper and is not used anywhere in the derivation above. Its paper-facing use of bilaterality and entireness in a terminal RH argument is reserved for H-PAP010. H-WIT002 and H-PAP010 are currently OPEN; see bug `#72f/2bff` and `docs/qa/2026-10-13-h-pap010-purpose-and-status-qa.md`. This pointer is included so that readers can locate the downstream target without mistaking it for an assumption or a closed conclusion of the present theorem.

H-PAP012: spectral operator-pair programme. The separately preregistered H-PAP012 paper studies the theta-spectral Hilbert–Pólya operator-pair question suggested by Hedenmalm's 2026 construction and audited in H-AUD098. It is downstream comparison/future work, not a theorem parent of the present bilateral-entire carrier. See bug `#72f/e205` and `docs/papers/h-pap012-hedenmalm-theta-spectral-hilbert-polya-preregistration.md`.

Downstream material deliberately excluded. The classical identification $Q_0^{\mathrm{HG}}(s) = \xi(s)$ is performed downstream inside Section 8; it is a recognition step, not a premise of the source construction or entire-transform proof. Repository material concerning zero location or RH lies farther downstream and is excluded from the theorem surface here. The comprehensive source-first RH manuscript H-PAP002 and the reserved terminal RH-closure manuscript H-PAP010 are likewise not theorem parents of this paper. This separation is what permits the present manuscript to state a narrow bilateral-entire theorem without importing an RH truth claim.

Frozen paper source ledger. The paper's `support/` directory contains the machine-readable authority manifest `source-manifest.json`. It records hashes for H-VER160, H-VER164, this paper's preregistration, the pre-LaTeX outline, and the controlling zero-cell Q&A. The manifest is a provenance lock: it does not replace the proofs in the theorem notes or the mathematics in this paper.

B Executable receipts and audit boundary

The repository contains exact symbolic receipts for the finite algebraic parts of both theorem owners. They are regression certificates, not a numerical search for Riemann zeros and not substitutes for the analytic arguments in the theorem notes.

H-VER160 receipt. The implementation module `c8_bilateral_entire_owner.py` in `src/d12rg_hypergroup/` and the regression `test_h_ver160_c8_bilateral_entire_owner.py` in `tests/` certify, among other exact identities, the distinction between original and dual zero floors, the finite Fourier reflection law, completed endpoint values, source retention, and the completed self-dual finite-vector functional equation. The closure receipt records that no external zeta, xi, Gamma, zero data, or RH truth value is consumed.

H-VER164 receipt. The implementation module `centered_bilateral_xi_source_bridge.py` in `src/d12rg_hypergroup/` and the regression `test_h_ver164_centered_bilateral_xi_source_bridge.py` in `tests/` certify the centered completion polynomial, the exact one-cell differential action, the zero value of the completed index-zero cell, parity preservation, and the classical kernel scale identity. These are symbolic identities. The rapid-decay and compact-uniform holomorphy argument proving the entire transform remains an analytic proof in H-VER164 and in Section 7.

Paper-specific structural checker. This paper has its own strict source-DAG contract, `section0-dag-roots.json`, in the paper's `support/` directory, with regression `test_h_pap005_section0_contract.py` in `tests/`. The certified sink requires the bilateral, entire, zero-safe, same-source-coordinate and no-external-ingress classifications. The positive-index display is typed a

lossy-readout firewall and has no downstream load-bearing edge. The contract also forbids an RH-equivalent tag at this paper's sink.

This executable boundary is deliberately modest. It checks exact algebra, provenance and paper structure; it does not turn numerical agreement into a theorem and does not move the RH-closure claim from its separate downstream owner into this paper.

C. Source-to-entire equation summary

Reader contract. This section is a compact audit/equation ledger; readers seeking the conventional mathematical exposition may begin with Section 2 and return here for the dependency summary. The paper starts from the complete bilateral Gaussian theta source and follows its centered differential completion. For repository provenance, these two ingredients are recorded under the internal theorem identifiers H-VER160 and H-VER164; those identifiers are audit labels, not additional mathematical assumptions. The original integer zero cell $n = 0$ is present and has value one before completion. The opposing half-shift completion annihilates that cell exactly; it does not delete it or divide by it. The resulting positive-index formula is therefore a compressed display of a retained full-lattice source. Entire continuation is a separate analytic theorem, proved directly from the completed kernel's two-sided decay; the resulting continuation agrees with the independently established completion recorded as H-VER160. Meromorphic zeta and Gamma factors, external zero tables, H-WIT002, and RH truth are not premises of the source-to-entire argument and are not conclusions of this paper.

Imported sources. The following equations are owned by another paper and imported, not redrawn; each item names its owner box.

- H-PAP001 [1], box front-pap001-archimedean-pi (owner H-VER077): $\Pi_{\mathrm{src}} := \lim_{n\to\infty} P_n^- = \lim_{n\to\infty} P_n^+$ and $\Pi_{\mathrm{src}} = \pi$. Ordinary π enters only through this endpoint recognition.
- H-PAP001 [1], box front-pap001-analytic-endpoints (owner H-VER155): $\exp z := \sum_{k=0}^{\infty} z^k/k!$.

Provenance: `H-VER160`.

$$\boxed{\Theta_0^{\mathrm{HG}}(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}, \qquad x > 0, \qquad \Theta_0^{\mathrm{HG}}(x)\Big|_{n=0} = 1.} \tag{1}$$

Provenance: `H-VER160`, `H-VER164-A`.

$$\boxed{x = e^{2u}, \qquad A_{\Xi}^{\mathrm{SCS}}(u) := e^{u/2}\Theta_0^{\mathrm{HG}}\left(e^{2u}\right), \qquad A_{\Xi}^{\mathrm{SCS}}(-u) = A_{\Xi}^{\mathrm{SCS}}(u).} \tag{2}$$

Provenance: `H-VER164-A`.

$$\boxed{\begin{gathered} D_u := \frac{d}{du}, \qquad K_{\xi}^{\mathrm{SCS}}(u) := \frac{1}{2}\left(D_u - \frac{1}{2}\right)\left(D_u + \frac{1}{2}\right) A_{\Xi}^{\mathrm{SCS}}(u), \\ K_{\xi}^{\mathrm{SCS}}(u) = \sum_{n\in\mathbb{Z}} \pi n^2 e^{5u/2}\left(2\pi n^2 e^{2u} - 3\right) e^{-\pi n^2 e^{2u}}, \\ K_{\xi}^{\mathrm{SCS}}(-u) = K_{\xi}^{\mathrm{SCS}}(u). \end{gathered}} \tag{3}$$

Provenance: `H-VER164-A`, `H-VER155`.

$$\boxed{e^{u/2}\Big|_{n=0} = e^{u/2} \neq 0.} \tag{4}$$

Provenance: `H-VER164-C`.

$$\boxed{\frac{1}{2}\left(\frac{d^2}{du^2} - \frac{1}{4}\right) e^{u/2} = 0, \qquad u \in \mathbb{C}.} \tag{5}$$

Thus the pre-completion zero floor is present, finite and nonzero, whereas the completed zero cell is present, finite and exactly zero.

Non-load-bearing compressed form: `H-VER164-B`.

$$\boxed{K_\xi^{\mathrm{SCS}}(u) = 2\sum_{n=1}^{\infty} \pi n^2 e^{5u/2}\left(2\pi n^2 e^{2u} - 3\right) e^{-\pi n^2 e^{2u}}.} \tag{6}$$

Equation (6) is numerically exact but forgets the explicit source ownership of the $n = 0$ zero cell. No load-bearing edge below depends on this compressed display.

Provenance: `H-VER160`.

$$\boxed{\begin{aligned} M_0(s) &:= \int_0^\infty [\Theta_0^{\mathrm{HG}}(x) - 1] x^{s/2-1}\, dx, \qquad \operatorname{Re} s > 1, \\ Q_0^{\mathrm{HG}}(s) &= \frac{1}{4} s(s-1) M_0(s), \qquad \operatorname{Re} s > 1, \\ Q_0^{\mathrm{HG}} &\text{ has the H-VER160 continuation } Q_0^{\mathrm{HG}} \in \mathcal{O}(\mathbb{C}). \end{aligned}} \tag{7}$$

The first line is the ordinary convergence-domain Mellin section. The second line defines the polynomial completion on that same half-plane; the third line records its entire H-VER160 continuation. From that point onward the symbol Q_0^{HG} denotes this unique entire continuation. It is not inferred from the raw integral without the endpoint completion.

Provenance: `H-VER160`, `H-VER164-B`.

$$\boxed{Q_0^{\mathrm{HG}}\left(\frac{1}{2} + \lambda\right) = \int_{-\infty}^{\infty} K_\xi^{\mathrm{SCS}}(u) e^{\lambda u}\, du, \qquad \operatorname{Re}\lambda > \frac{1}{2}.} \tag{8}$$

Equation (8) is already *bilateral* in the logarithmic coordinate: its integration domain is $u \in \mathbb{R}$. At this stage the identity is still proved only for $\operatorname{Re}\lambda > \frac{1}{2}$. The raw Mellin section M_0 is one-sided in $x > 0$ and has a meromorphic continuation with explicit endpoint poles; it is the completed section Q_0^{HG} that is entire. The next equation is therefore the distinct analytic step that proves that this same bilateral integral converges for every $\lambda \in \mathbb{C}$, defines an entire function there, and extends the identity from its initial half-plane.

Main theorem: `H-VER160`, `H-VER164-B`.

$$\boxed{\begin{aligned} Q_0^{\mathrm{SCS}}\left(\frac{1}{2}+\lambda\right) &:= \int_{-\infty}^{\infty} K_\xi^{\mathrm{SCS}}(u) e^{\lambda u}\, du, \qquad \lambda \in \mathbb{C}, \\ Q_0^{\mathrm{SCS}}\left(\frac{1}{2}+\lambda\right) &= Q_0^{\mathrm{HG}}\left(\frac{1}{2}+\lambda\right), \qquad \lambda \in \mathbb{C}, \end{aligned}} \tag{9}$$

and both sides of Equation (9) are entire functions of λ. The extension from Equation (8) is furnished directly by the entire whole-line integral, using rapid two-sided decay and compact-uniform domination; it does not follow merely from the zero value in Equation (5). Uniqueness of analytic continuation identifies this with the independently owned H-VER160 completion.

Provenance: `H-VER164-B`.

$$\boxed{\begin{aligned} \Xi_{\mathrm{HG}}(t) &:= Q_0^{\mathrm{HG}}\left(\frac{1}{2}+it\right) \\ &= \int_{-\infty}^{\infty} K_\xi^{\mathrm{SCS}}(u) e^{itu}\, du \\ &= 2\int_{0}^{\infty} K_\xi^{\mathrm{SCS}}(u)\cos(tu)\, du, \qquad t \in \mathbb{C}. \end{aligned}} \tag{10}$$

Downstream classical recognition: `H-VER164-A` (the `H-VER164` S8 cross-check).

$$\boxed{\begin{aligned} \Phi_{\mathrm{RH}}(v) &:= \sum_{n=1}^{\infty} \left(2\pi^2 n^4 e^{9v} - 3\pi n^2 e^{5v}\right) e^{-\pi n^2 e^{4v}}, \\ K_\xi^{\mathrm{SCS}}(2v) &= 2\Phi_{\mathrm{RH}}(v). \end{aligned}} \tag{11}$$

The equality is an exact scale cross-check. Since $K_\xi^{\mathrm{SCS}}(-u) = K_\xi^{\mathrm{SCS}}(u)$, it also implies $\Phi_{\mathrm{RH}}(-v) = \Phi_{\mathrm{RH}}(v)$: the classical kernel is even. Its usual half-line cosine representation is therefore a symmetry compression of a bilateral whole-line kernel. Historical Xi and De Bruijn–Newman references [4, 5, 6, 10] attach the conventional name downstream; they are not used to prove the preceding source route.

This paper's theorem package: `H-VER160`, `H-VER164-B`.

$$\boxed{\begin{array}{c}\text{complete bilateral theta source with retained zero floor}\\ \Longrightarrow \text{regular opposing-half-shift zero-cell annihilation}\\ \Longrightarrow \text{bilateral, zero-safe, entire } \Xi_{\mathrm{HG}} \text{ carrier,}\\ \text{with no RH truth statement proved or assumed in this paper.}\\ \Xi_{\mathrm{HG}}(t)=\displaystyle\int_{-\infty}^{\infty} K_{\xi}^{\mathrm{SCS}}(u)e^{itu}\,du, \qquad t\in\mathbb{C}.\end{array}} \tag{12}$$

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