First-Principles Derivation of π from the Rank-One Chebyshev Rail of a Ten-Sector Verlinde–Weyl Hypergroup

Luke Kenneth Casson Leighton

8 September 2026

Abstract

We derive the geometric circle constant from the canonical rank-one Chebyshev rail contained in a finite rank-ten Verlinde probability hypergroup, without taking π, e, exponential or logarithmic coordinates, trigonometric functions, or a remembered series for π as upstream data. The finite parent has already been reconstructed exactly as the level-two $A_3^{(1)}$ Weyl-character generalized-Chebyshev quotient. Its order-four simple current supplies an intrinsic four-ality coordinate, and the inversion-even restriction of that coordinate is the ordinary first-kind Chebyshev hypergroup. The higher-rank data establish canonical ancestry of that rail; the load-bearing Archimedean proof after restriction is rank one. The source product therefore fixes $T_2(x) = 2x^2 - 1$. Iterating the positive inverse branch gives a purely algebraic half-step tower. The half-step is proved to be the Euclidean angle bisector, and the resulting inscribed and circumscribed polygon ratios form an exact monotone squeeze. Their common Archimedean completion defines a source scalar Π_{src}. By the geometric definition of the circle constant, $\Pi_{\mathrm{src}} = \pi$. The same completed half-step tower also produces the dyadic rotation group directly. Its source arclength normalization forces the differential pair $C' = -S$, $S' = C$, hence the sine and cosine series, and then forces the entire exponential series and its multiplicative law. Euler's formula is therefore a downstream theorem, not an input. The same theorem yields Viète's product as a source half-step identity. A second source expansion gives the Leibniz series, and a third consequence identifies Π_{src} as the normalizer of the standard first-kind Chebyshev arcsine spectral measure. H-VER080 then proves that the same non-circular source scalar normalizes all four geometric C_4 quadrants and every reciprocal/conjugate off-boundary branch projection. The paper separates classical literature ingredients from the reversed project theorem: the novel claim is not any one classical formula for π, but the exact source ancestry from the ten-sector Verlinde/Weyl-generalised-Chebyshev hypergroup to the required infinite Archimedean completion.

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0. Proof constitution and Archimedean source DAG

This Section 0 is the paper's complete front-facing proof contract. It uses four distinct roles throughout: a canonical HG theorem owner, a local certificate derived from that owner, any reconstruction needed before a richer owner-level claim, and the downstream consumer of the result. A paper equation, executable receipt, or literature identity may certify or compare a claim; none becomes a canonical HG source merely by being useful.

The rank-ten to rank-one step is deliberately typed as a canonical H-VER072-owned subobject selection. The paper never reconstructs the full rank-ten object from the one-variable Chebyshev rail and never needs to do so: all Archimedean claims after that point are rail-typed. Consequently no full-ten information is resurrected from the reduced carrier. The public closed package below returns only claims independently owned by H-VER077/H-VER078/H-VER080/H-VER155.

One visible side route remains non-certifying. The standard Chebyshev spectral measure is literature-owned even though H-VER079 owns its source normalizer. By contrast, H-VER155 now owns the source sine/cosine/exponential packet, so that packet lies on the certified sink path.

Provenance: `H-VER064`, `H-VER068`, `H-VER070`, `H-VER071`.

$$\boxed{\begin{aligned} H_{10} \cong \operatorname{Norm} \mathcal{V}_2(A_3^{(1)}), \qquad |H_{10}| = 10, \\ g_4^4 = 1. \end{aligned}} \tag{1}$$

Provenance: `H-VER072`.

$$\boxed{\begin{aligned} E_n &:= \frac{g_4^n + g_4^{-n}}{2}, \\ E_m E_n &= \tfrac{1}{2}\big(E_{m+n} + E_{|m-n|}\big), \\ T_2(x) &= 2x^2 - 1. \end{aligned}} \tag{2}$$

Provenance: `H-VER077`.

$$\boxed{\begin{gathered} R := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad R^4 = I, \\ x_n\big|_{n=0} = 0, \qquad x_{n+1} = \sqrt{\frac{1+x_n}{2}}, \\ T_2(x_{n+1}) = x_n, \\ y_n := \sqrt{1-x_n^2}, \qquad y_n = 2x_{n+1}y_{n+1}, \\ v_{n+1} = \frac{e+v_n}{\|e+v_n\|}, \qquad v_n = (x_n, y_n), \\ e = (1,0). \end{gathered}} \tag{3}$$

Provenance: `H-VER077`.

$$\boxed{\begin{gathered} \prod_{k=1}^{N} x_k = \frac{1}{2^N y_N}, \\ P_n^- := 2^{n+2} y_{n+1} = \frac{2}{\prod_{k=1}^{n+1} x_k}, \\ P_n^+ := \frac{P_n^-}{x_{n+1}}. \end{gathered}} \tag{4}$$

Provenance: `H-VER077`.

$$\boxed{\begin{gathered} P_n^- \nearrow \Pi_{\text{src}}, \qquad P_n^+ \searrow \Pi_{\text{src}}, \qquad \frac{P_n^+}{P_n^-} = \frac{1}{x_{n+1}} \longrightarrow 1, \\ \Pi_{\text{src}} := \lim_{n\to\infty} P_n^- = \lim_{n\to\infty} P_n^+, \\ \boxed{\Pi_{\text{src}} = \pi}. \end{gathered}} \tag{5}$$

Provenance: `H-VER077`.

$$\boxed{\frac{2}{\Pi_{\text{src}}} = \prod_{n=1}^{\infty} x_n, \qquad x_{n+1} = \sqrt{\frac{1+x_n}{2}}, \quad x_0 = 0.} \tag{6}$$

Provenance: `H-VER078`.

$$\boxed{\begin{gathered} \gamma_0(t) := \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right), \qquad 0 \le t \le 1, \\ \|\gamma_0'(t)\| = \frac{2}{1+t^2}, \qquad \Pi_{\text{src}} = 4\int_0^1 \frac{dt}{1+t^2}. \end{gathered}} \tag{7}$$

Provenance: `H-VER078`.

$$\boxed{\Pi_{\text{src}} = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.} \tag{8}$$

Provenance: `H-VER080`.

$$\boxed{\begin{gathered}\gamma_k(t) := R^k\gamma_0(t), \qquad k = 0,1,2,3,\\ \|\gamma_k'(t)\| = \frac{2}{1+t^2}, \qquad \operatorname{length}(Q_k) = \frac{\Pi_{\mathrm{src}}}{2},\\ \operatorname{length}(S^1) = 2\Pi_{\mathrm{src}}.\end{gathered}} \tag{9}$$

Provenance: `H-VER079`; literature-owned measure theorem.

$$\boxed{\begin{gathered}d\mu_{\mathrm{Ch}}(x) = \frac{dx}{\Pi_{\mathrm{src}}\sqrt{1-x^2}}, \qquad -1 < x < 1,\\ \text{measure shape external; normalizer HG-owned.}\end{gathered}} \tag{10}$$

Firewall: the displayed measure is useful downstream, but its literature-owned shape cannot become an HG source premise.

Provenance: `H-VER155`; closed source analytic packet.

$$\boxed{\begin{gathered}C_{\mathrm{src}}'(s) = -S_{\mathrm{src}}(s), \qquad S_{\mathrm{src}}'(s) = C_{\mathrm{src}}(s),\\ \operatorname{Exp}_{\mathrm{src}}(z) := \sum_{k=0}^{\infty}\frac{z^k}{k!}, \qquad \operatorname{Exp}_{\mathrm{src}}(z+w) = \operatorname{Exp}_{\mathrm{src}}(z)\operatorname{Exp}_{\mathrm{src}}(w),\\ \operatorname{Exp}_{\mathrm{src}}(is) = C_{\mathrm{src}}(s) + iS_{\mathrm{src}}(s),\\ \text{H-VER155 closed source analytic packet.}\end{gathered}} \tag{11}$$

Ownership: H-VER155 is now the canonical theorem owner. The displayed paper equations and executable receipt certify that theorem; they do not replace its semantic ownership.

Provenance: `H-VER155`; conventional endpoint aliases.

$$\boxed{\begin{gathered}\exp z := \sum_{k=0}^{\infty}\frac{z^k}{k!}, \qquad \cos s := \frac{\exp(is)+\exp(-is)}{2}, \qquad \sin s := \frac{\exp(is)-\exp(-is)}{2i},\\ \operatorname{Exp}_{\mathrm{src}} = \exp, \qquad C_{\mathrm{src}} = \cos, \qquad S_{\mathrm{src}} = \sin,\\ \text{exact H-VER155 endpoint aliases after source construction (H-VER155 S7).}\end{gathered}} \tag{12}$$

Provenance: these conventional names are introduced only at the endpoint of the H-VER155 source construction. They are not foundation symbols or remembered analytic premises.

Provenance: `H-VER077`, `H-VER078`, `H-VER080`, `H-VER155`.

$$\boxed{\begin{gathered}\Pi_{\mathrm{src}} = \pi, \qquad \frac{2}{\Pi_{\mathrm{src}}} = \prod_{n=1}^{\infty} x_n,\\ \Pi_{\mathrm{src}} = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1},\\ \operatorname{length}(\gamma_k) = \frac{\Pi_{\mathrm{src}}}{2} \quad (k = 0,1,2,3),\\ \operatorname{Exp}_{\mathrm{src}} = \exp, \qquad C_{\mathrm{src}} = \cos, \qquad S_{\mathrm{src}} = \sin,\\ \text{no external theorem lies in this certified closure.}\end{gathered}} \tag{13}$$

1 Scope and theorem boundary

The declared source is the exact ten-sector Verlinde hypergroup already owned by H-VER064–H-VER072. The paper may use only the following closed finite parents:

H-VER064	rank-ten commutative Verlinde probability hypergroup;
H-VER068	exact $A_3^{(1)}$ level-two Weyl-generalised-Chebyshev isomorphism;
H-VER070	entrywise Weyl/Verlinde/hypergroup Fourier identification;
H-VER071	intrinsic order-four simple current and $\mathbb{Z}/4\mathbb{Z}$ four-ality;
H-VER072	first- and second-kind Chebyshev rails as exact restrictions of that four-ality calculus.

The target is a first-principles construction of the geometric circle constant. The following are forbidden as upstream premises:

$$\pi, \qquad e, \qquad \exp, \qquad \log, \qquad \sin, \cos, \tan, \qquad \arctan .$$

No De Vries–Jordan–Jacobi (DVJ), fine-structure, zeta-function, or Riemann Hypothesis datum is permitted anywhere in the proof.

The theorem is therefore not a statement that a finite algebraic object can produce a transcendental number by finitely many algebraic operations. The finite algebra supplies a canonical refinement rule; the transcendental readout appears only after an explicitly declared infinite Archimedean completion.

2 The rank-ten source is already a Weyl-generalised-Chebyshev hypergroup

Let H_{10} denote the normalized ten-sector fusion hypergroup. H-VER068 reconstructs the same multiplication independently from the level-two $A_3^{(1)}$ Weyl-character quotient and proves

$$\boxed{H_{10} \cong \operatorname{Norm} \mathcal{V}_2(A_3^{(1)}).}$$

This is stronger than a rank or spectrum match: all one hundred fusion products and all one hundred normalized probability products agree exactly.

H-VER070 then identifies the finite harmonic analyses entry-by-entry. If λ, μ are the ten dominant level-two A_3 weights, with Verlinde points built from the principal 24th root of unity, then

$$\boxed{\chi_\lambda^{A_3}(\mu) = \frac{S_{\lambda\mu}}{S_{0\mu}}.}$$

Thus the Verlinde Fourier table and the A_3 Weyl-character table are the same exact finite object, column for column. The affine half-turn J^2 of the point labels appears only under the Galois conjugation sending that root to its fifth power.

The order-four simple current g is affine Dynkin rotation. H-VER071 identifies its Fourier character with the intrinsic four-ality coordinate $q(\mu) \in \mathbb{Z}/4\mathbb{Z}$. Writing

$$z_\mu = i^{q(\mu)},$$

the even and odd reciprocal coordinates are

$$E_n(\mu) = \frac{z_\mu^n + z_\mu^{-n}}{2}, \qquad J_n(\mu) = \frac{z_\mu^n - z_\mu^{-n}}{2i}.$$

H-VER072 proves that these are precisely the first- and second-kind Chebyshev restrictions of the full ten-point Weyl Fourier calculus.

This establishes the source chain

Verlinde $10 \times 10 = A_3$ Weyl-generalised-Chebyshev hypergroup $\supset$ canonical C_4/A_1 Chebyshev rail.

The rank-one rail is a restriction of the full rank-ten object, not a replacement for it. Conversely, once H-VER072 has canonically selected that rail, the polygon proof below uses only its one-variable Chebyshev recurrence.

3 The source multiplication fixes the doubling map

The first-kind radial operators are

$$E_n = \frac{N_g^n + N_g^{-n}}{2}.$$

Because $g^4 = 1$, direct multiplication gives the Chebyshev hypergroup law

$$\boxed{E_m E_n = \frac{1}{2}(E_{m+n} + E_{|m-n|}).}$$

In particular,

$$E_1^2 = \frac{1}{2}(E_0 + E_2).$$

With $E_0 = 1$ and $x = E_1$ this is

$$\boxed{T_2(x) = 2x^2 - 1.}$$

No angular coordinate is required to obtain this polynomial. It is fixed by the source hypergroup multiplication itself.

4 The positive algebraic half-step tower

The real C_4 source has the canonical quarter-turn representation

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad R^4 = I,$$

whose orbit through $(1, 0)$ is

$$(1,0),\ (0,1),\ (-1,0),\ (0,-1).$$

The first source quarter selects the nonnegative inverse branch of the Chebyshev doubling map. Define

$$x_0 = 0, \qquad \boxed{x_{n+1} = \sqrt{\frac{1 + x_n}{2}}.}$$

Then

$$T_2(x_{n+1}) = x_n$$

exactly at every finite stage.

Put

$$y_n = \sqrt{1 - x_n^2}.$$

The half-step equation gives

$$x_{n+1}^2 = \frac{1 + x_n}{2}, \qquad y_{n+1}^2 = \frac{1 - x_n}{2}.$$

Since all quantities are nonnegative on the chosen quarter,

$$\boxed{y_n = 2x_{n+1}y_{n+1}.}$$

Write $v_n = (x_n, y_n)$ and $e = (1, 0)$. Directly,

$$\|e + v_n\|^2 = (1 + x_n)^2 + y_n^2 = 2(1 + x_n) = (2x_{n+1})^2.$$

The positive branch therefore satisfies

$$\boxed{v_{n+1} = \frac{e + v_n}{\|e + v_n\|}.}$$

Thus v_{n+1} is the Euclidean angle bisector of the first source quarter, so repeated half-steps really produce equal subarcs and equal chords without introducing a trigonometric coordinate.

The same elementary identities provide the finite telescope underlying the Archimedean construction.

5 Finite polygons and the Viète telescope

Starting from $y_0 = 1$, the previous identity gives

$$\boxed{\prod_{k=1}^{N} x_k = \frac{1}{2^N y_N}.}$$

Now consider the chord from $(1, 0)$ to (x_n, y_n). Its squared length is

$$(1 - x_n)^2 + y_n^2 = 2(1 - x_n) = 4y_{n+1}^2,$$

so the positive chord length is $2y_{n+1}$.

After n half-step refinements, the four source quarters contain 2^n equal chords each. The corresponding inscribed regular polygon therefore has $4 \cdot 2^n$ sides. Its perimeter divided by the diameter of the unit circle is

$$P_n^- = 2^{n+2} y_{n+1}.$$

By the finite telescope,

$$\boxed{P_n^- = \frac{2}{\prod_{k=1}^{n+1} x_k}.}$$

Thus the finite Viète product and the finite polygonal readout are exactly the same source object at every refinement level.

The corresponding circumscribed polygon ratio is

$$P_n^+ = \frac{P_n^-}{x_{n+1}}.$$

Both quantities are obtained without any transcendental constant.

6 Archimedean completion and the source circle constant

The recurrence implies

$$1 - x_{n+1} = \frac{1 - x_n}{2(1 + x_{n+1})} \leq \frac{1 - x_n}{2},$$

so $x_n \to 1$. The common-limit claim is an exact squeeze, not merely a numerical observation. Put $x = x_{n+1}$, $L = P_n^-$, and $x' = x_{n+2} = \sqrt{(1+x)/2}$. Then

$$P_{n+1}^- = \frac{L}{x'} \geq L = P_n^-$$

because $x' \leq 1$, while

$$P_n^+ = \frac{L}{x}, \qquad P_{n+1}^+ = \frac{2L}{1+x},$$

and hence

$$\boxed{\frac{P_n^+ - P_{n+1}^+}{L} = \frac{1-x}{x(1+x)} \geq 0.}$$

Thus the inscribed ratios increase and the circumscribed ratios decrease. Finally,

$$\boxed{\frac{P_n^+}{P_n^-} = \frac{1}{x_{n+1}} \longrightarrow 1,}$$

so the two monotone sequences have one common limit.

Define the source Archimedean scalar by the common polygon limit

$$\boxed{\Pi_{\text{src}} := \lim_{n\to\infty} P_n^- = \lim_{n\to\infty} P_n^+.}$$

This is the first operation in the proof that leaves finite algebra. The finite source fixes the entire refining sequence; the real completion supplies its limiting scalar.

Theorem 6.1 (H-VER077)**.** *The rank-ten Verlinde/Weyl-generalised-Chebyshev hypergroup canonically contains the rank-one Chebyshev rail whose positive inverse branch produces an Archimedean scalar* Π_{src} *equal to the limiting circumference/diameter ratio of dyadically refined regular polygons. Hence, under the geometric definition of the circle constant,*

$$\boxed{\Pi_{\text{src}} = \pi.}$$

No value or analytic representation of π *is used upstream.*

The theorem also gives Viète's product as a corollary:

$$\boxed{\frac{2}{\Pi_{\text{src}}} = \prod_{n=1}^{\infty} x_n, \qquad x_{n+1} = \sqrt{\frac{1+x_n}{2}}, \quad x_0 = 0.}$$

Only after the theorem endpoint may Π_{src} be renamed π.

7 Source construction of sine, cosine, and exponential

The same half-step tower contains more than the scalar Π_{src}. It also contains the complete dyadic rotation system, so no exponential or trigonometric function has to be imported in order to parametrize the circle.

Define

$$R_n = \begin{pmatrix} x_n & -y_n \\ y_n & x_n \end{pmatrix}, \qquad n \geq 0.$$

The source initial condition gives

$$R_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = R, \qquad R^4 = I.$$

Using

$$x_{n+1}^2 - y_{n+1}^2 = x_n, \qquad 2x_{n+1}y_{n+1} = y_n,$$

one obtains directly

$$\boxed{R_{n+1}^2 = R_n.}$$

Thus the positive Chebyshev inverse branch supplies a canonical nested system of algebraic square roots of the source quarter-turn itself.

The source quarter has length $\Pi_{\mathrm{src}}/2$. Put

$$s_n := \frac{\Pi_{\mathrm{src}}}{2^{n+1}}.$$

On the dyadic subgroup

$$\mathcal{D} = \bigcup_{n \geq 0} s_n \mathbb{Z} \subset \mathbb{R}$$

define

$$\boxed{U(ms_n) := R_n^m, \qquad m \in \mathbb{Z}.}$$

The relation $R_{n+1}^2 = R_n$ proves that this definition is independent of which refinement level represents the same dyadic source length. It also gives

$$U(a+b) = U(a)U(b), \qquad U(2\Pi_{\mathrm{src}}) = I, \qquad U(\Pi_{\mathrm{src}}/2) = R$$

on $\mathcal{D}$. The same chord-versus-arc squeeze used above to define Π_{src} makes this dyadic map uniformly continuous in source arclength. Its Archimedean completion therefore gives a unique continuous homomorphism

$$U : \mathbb{R}/(2\Pi_{\mathrm{src}}\mathbb{Z}) \longrightarrow SO(2).$$

Write its entries as

$$\boxed{U(s) = \begin{pmatrix} C_{\mathrm{src}}(s) & -S_{\mathrm{src}}(s) \\ S_{\mathrm{src}}(s) & C_{\mathrm{src}}(s) \end{pmatrix}.}$$

These functions are definitions coming from the completed hypergroup half-step, not renamed trigonometric premises. Matrix multiplication gives immediately

$$\boxed{\begin{aligned} C_{\mathrm{src}}(s+t) &= C_{\mathrm{src}}(s)C_{\mathrm{src}}(t) - S_{\mathrm{src}}(s)S_{\mathrm{src}}(t), \\ S_{\mathrm{src}}(s+t) &= S_{\mathrm{src}}(s)C_{\mathrm{src}}(t) + C_{\mathrm{src}}(s)S_{\mathrm{src}}(t), \end{aligned}}$$

together with

$$C_{\mathrm{src}}(s)^2 + S_{\mathrm{src}}(s)^2 = 1, \qquad C_{\mathrm{src}}(0) = 1, \qquad S_{\mathrm{src}}(0) = 0,$$

and

$$C_{\mathrm{src}}(\Pi_{\mathrm{src}}/2) = 0, \qquad S_{\mathrm{src}}(\Pi_{\mathrm{src}}/2) = 1.$$

The infinitesimal normalization is also forced by the same polygon squeeze. At the source mesh point s_n,

$$C_{\mathrm{src}}(s_n) = x_n, \qquad S_{\mathrm{src}}(s_n) = y_n.$$

Since

$$P_{n-1}^{-} = 2^{n+1} y_n \quad \text{and} \quad P_{n-1}^{-} \longrightarrow \Pi_{\mathrm{src}},$$

one has

$$\frac{S_{\mathrm{src}}(s_n)}{s_n} = \frac{P_{n-1}^{-}}{\Pi_{\mathrm{src}}} \longrightarrow 1.$$

Also

$$1 - x_n = \frac{y_n^2}{1 + x_n},$$

so

$$\frac{C_{\mathrm{src}}(s_n) - 1}{s_n} \longrightarrow 0.$$

The chord-versus-arc estimate also gives a local Lipschitz bound on the dyadic orbit. Before passing to arbitrary real increments, the displayed mesh-generator limit must be extended to every dyadic increment. For $d = ms_n$ with $m > 0$, the group law gives

$$U(ms_n) - I = \sum_{j=0}^{m-1} U(js_n)(U(s_n) - I),$$

so

$$\boxed{\frac{U(ms_n) - I}{ms_n} = \frac{1}{m} \sum_{j=0}^{m-1} U(js_n) \frac{U(s_n) - I}{s_n}.}$$

For an arbitrary sequence $d_r = m_r s_{n_r} \to 0$, refinement-independence allows representatives with $n_r \to \infty$. Since $|js_{n_r}| \le |d_r|$, continuity at zero gives

$$\max_{0 \le j < m_r} \|U(js_{n_r}) - I\| \longrightarrow 0.$$

Thus the average above tends uniformly to I, while the mesh-generator quotient tends to R. Hence

$$\boxed{\frac{U(d) - I}{d} \longrightarrow R \qquad (d \in \mathcal{D},\ d \to 0).}$$

Negative dyadic increments follow from the group inverse, or from the same telescoping argument backwards.

Now let arbitrary $h \to 0$. Choose a dyadic d_h with $|h - d_h| = o(h^2)$. Continuity of the completion and the Lipschitz bound make the difference between the h and d_h difference quotients tend to zero. The arbitrary-dyadic limit therefore passes to the full Archimedean completion, and

$$U'(0) = R.$$

Differentiating the already-proved group law $U(s + t) = U(s)U(t)$ at $t = 0$ gives

$$\boxed{U'(s) = U(s)R = RU(s),}$$

and therefore

$$\boxed{C'_{\mathrm{src}}(s) = -S_{\mathrm{src}}(s), \qquad S'_{\mathrm{src}}(s) = C_{\mathrm{src}}(s).}$$

No sine, cosine, exponential, logarithm, or Euler identity has entered this argument.

Put

$$E_{\mathrm{src}}(s) := C_{\mathrm{src}}(s) + iS_{\mathrm{src}}(s).$$

The source differential pair is exactly

$$E'_{\mathrm{src}}(s) = iE_{\mathrm{src}}(s), \qquad E_{\mathrm{src}}(0) = 1.$$

Thus, recursively,

$$E_{\mathrm{src}}^{(k)}(s) = i^k E_{\mathrm{src}}(s).$$

Since $U(s) \in SO(2)$, one has $|E_{\mathrm{src}}(s)| = 1$. Taylor's remainder at zero is therefore bounded by

$$|R_N(s)| \leq \frac{|s|^{N+1}}{(N+1)!},$$

which tends to zero. Hence the source ODE itself forces

$$E_{\mathrm{src}}(s) = \sum_{k=0}^{\infty} \frac{(is)^k}{k!}.$$

Taking real and imaginary parts gives

$$\boxed{C_{\mathrm{src}}(s) = \sum_{k=0}^{\infty} \frac{(-1)^k s^{2k}}{(2k)!},}$$

$$\boxed{S_{\mathrm{src}}(s) = \sum_{k=0}^{\infty} \frac{(-1)^k s^{2k+1}}{(2k+1)!}.}$$

These familiar series have therefore been derived from the source half-step and its Archimedean completion rather than supplied as definitions.

Removing the source-line factor i^k gives the unique entire continuation forced by the same coefficient recurrence. Define

$$\boxed{\mathrm{Exp}_{\mathrm{src}}(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}.}$$

The ratio test proves convergence for every $z \in \mathbb{C}$; no exponential function has been assumed. The coefficient recurrence is

$$(k+1)a_{k+1} = a_k, \qquad a_0 = 1,$$

so $a_k = 1/k!$. The binomial theorem and the absolutely convergent Cauchy product then give

$$\boxed{\mathrm{Exp}_{\mathrm{src}}(z+w) = \mathrm{Exp}_{\mathrm{src}}(z)\,\mathrm{Exp}_{\mathrm{src}}(w).}$$

On the compact source line the two constructions coincide:

$$\boxed{\mathrm{Exp}_{\mathrm{src}}(is) = C_{\mathrm{src}}(s) + iS_{\mathrm{src}}(s).}$$

Consequently

$$\boxed{C_{\mathrm{src}}(s) = \frac{\mathrm{Exp}_{\mathrm{src}}(is) + \mathrm{Exp}_{\mathrm{src}}(-is)}{2},}$$

$$\boxed{S_{\mathrm{src}}(s) = \frac{\mathrm{Exp}_{\mathrm{src}}(is) - \mathrm{Exp}_{\mathrm{src}}(-is)}{2i}.}$$

Euler's formula is thus a theorem at the end of the construction. Only thereafter may one identify

$$\boxed{\mathrm{Exp}_{\mathrm{src}} = \exp, \qquad C_{\mathrm{src}} = \cos, \qquad S_{\mathrm{src}} = \sin,}$$

because the conventional exponential is defined by the same entire power series, while conventional cosine and sine are the same exponential combinations displayed above. In particular

$$e_{\mathrm{src}} := \mathrm{Exp}_{\mathrm{src}}(1)$$

is a downstream source scalar. No logarithm or logarithmic branch has been used or claimed.

This gives the explicit audit direction

Chebyshev hypergroup → positive algebraic half-step → dyadic rotation group → Archimedean one-paramet

The only non-finite operation is the same Archimedean completion already required by H-VER077 to obtain Π_{src}.

8 C4-equivariant completion of all four quadrants

H-VER080 removes a possible branch ambiguity left implicit by the first-quarter construction. Let

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad R^4 = I,$$

and let γ_0 be the rational first-quarter parametrization used below. Define

$$\gamma_k(t) = R^k \gamma_0(t), \qquad k = 0, 1, 2, 3.$$

Because $R^T R = I$, every quarter satisfies exactly

$$\|\gamma_k(t)\|^2 = 1, \qquad \|\gamma_k'(t)\|^2 = \frac{4}{(1+t^2)^2}.$$

Thus all four geometric quadrants are isometric copies of the same source quarter before any angular or trigonometric coordinate is introduced.

Writing

$$L_{\mathrm{src}} = \int_0^1 \frac{dt}{1+t^2},$$

H-VER078 gives $\Pi_{\mathrm{src}} = 4L_{\mathrm{src}}$. Each quadrant therefore has arc length

$$\frac{\Pi_{\mathrm{src}}}{2},$$

the full unit circle has length $2\Pi_{\mathrm{src}}$, and the normalized boundary arclength measure

$$d\nu = \frac{ds}{2\Pi_{\mathrm{src}}}$$

has mass $1/4$ on every quadrant. There is consequently one C_4-equivariant Archimedean boundary normalizer, not four independent choices.

The same theorem also controls a generic nonzero off-boundary complex packet. Write $q = ru$ with $r > 0$ and $|u| = 1$, without choosing an angle. Then direct algebra gives

$$\boxed{q = ru, \qquad q^\sharp = r^{-1}u, \qquad \overline{q} = r\overline{u}, \qquad q^{-1} = r^{-1}\overline{u}.}$$

Hence boundary projection $\nu(z) = z/|z|$ satisfies

$$\boxed{\nu(q) = \nu(q^\sharp) = u, \qquad \nu(\overline{q}) = \nu(q^{-1}) = \overline{u}.}$$

Reciprocal sharp changes only radial sheet; conjugation and inversion reflect boundary orientation. All four labeled branches therefore use the same source-normalized C_4 boundary.

The theorem deliberately does not identify the four reciprocal/conjugate branch labels with the four geometric quadrants. Those are respectively a commuting $C_2 \times C_2$ branch grammar and a cyclic C_4 geometry. H-VER080 proves their boundary compatibility, not equality of the two actions.

9 A second source readout: the Leibniz series

H-VER078 derives a second representation of the same already-defined source constant. The first source quarter has the rational parametrization

$$\gamma(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right), \qquad 0 \le t \le 1.$$

Direct algebra gives $x(t)^2 + y(t)^2 = 1$, and direct differentiation gives

$$x'(t)^2 + y'(t)^2 = \frac{4}{(1+t^2)^2}.$$

Hence one source quarter has length

$$2\int_0^1 \frac{dt}{1+t^2},$$

and therefore

$$\Pi_{\mathrm{src}} = 4\int_0^1 \frac{dt}{1+t^2}.$$

For finite N the exact rational identity

$$\frac{1}{1+t^2} = \sum_{n=0}^{N} (-1)^n t^{2n} + \frac{(-1)^{N+1} t^{2N+2}}{1+t^2}$$

has integrated remainder bounded by $1/(2N+3)$. Consequently

$$\boxed{\Pi_{\mathrm{src}} = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.}$$

The finite geometric-series identity already forces the alternating sign. Independently, the same sign is the real orientation character on the two odd four-ality classes 1 and 3:

$$\chi_4(2n+1) = (-1)^n.$$

Thus the source orientation identifies and reproduces the algebraically forced sign sequence. The familiar Leibniz series is a downstream expansion of the same source quarter rather than a formula selected because its sum was known.

10 A third consequence: Chebyshev spectral normalization

The first-kind Chebyshev polynomial hypergroup has the standard arcsine orthogonality/Plancherel density

$$d\mu(x) = \frac{1}{C}\frac{dx}{\sqrt{1-x^2}}, \qquad -1 < x < 1.$$

This standard measure theorem remains literature-owned. What changes after H-VER077 is the status of its normalizing constant.

On $0 \le x \le 1$, use the same rational substitution

$$x = \frac{1-t^2}{1+t^2}, \qquad \sqrt{1-x^2} = \frac{2t}{1+t^2}.$$

Then

$$\frac{-dx}{\sqrt{1-x^2}} = \frac{2\,dt}{1+t^2}.$$

By symmetry,

$$C = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = 4\int_{0}^{1} \frac{dt}{1+t^2} = \Pi_{\mathrm{src}}.$$

Hence

$$\boxed{d\mu(x) = \frac{dx}{\Pi_{\mathrm{src}}\sqrt{1-x^2}}.}$$

The source-derived scalar is therefore not only a geometric constant. It is also the harmonic-analysis normalizer of the canonical first-kind Chebyshev rail inside the rank-ten source.

11 In the literature

The literature contains essentially all of the individual ingredients, but normally with the direction of dependence reversed.

Verlinde fusion. Verlinde showed that modular transformation data determine the fusion rules of rational conformal field theory [1]. The present paper does not re-prove that general theorem; H-VER064 is the project-specific rank-ten instance.

Chebyshev and fusion. Furlan, Hadjiivanov, and Todorov explicitly use Chebyshev polynomials in an $su(2)$ fusion-ring setting [2]. Tsurii gives the product law

$$T_m T_n = \frac{1}{2}T_{|m-n|} + \frac{1}{2}T_{m+n}$$

for the first-kind Chebyshev hypergroup and treats its relation to fusion-rule algebras [3].

Weyl-generalised Chebyshev theory. Hoffman and Withers construct generalized Chebyshev polynomials associated with affine Weyl groups as stretch/fold polynomial maps; rank one reduces, up to conjugation, to the ordinary Chebyshev construction [4]. Nesterenko, Patera, and Tereszkiewicz identify the $SU(2)$ orbit functions with classical Chebyshev polynomials and the $SU(n)$ orbit functions with multivariable generalizations [5]. Hubert, Metzlaff, and Riener identify compact-torus Weyl orbit spaces as orthogonality regions of generalized Chebyshev polynomials [6].

Viète and polygon refinement. Viète's product is classical. Moreno and Garcia-Caballero review its status as the first recorded infinite product, its polygonal origin, and its relation to iterated doubling/half-angle formulas [7]. The present result reverses this use: the half-step map is first obtained from the hypergroup product, and its polygon limit is only then identified as the circle constant.

Chebyshev spectral measure. Polynomial-hypergroup literature treats the first-kind Chebyshev hypergroup and its harmonic analysis; Lasser is a standard reference in this setting [8]. The conventional arcsine density is usually written with $1/\pi$ already present. H-VER079 instead supplies that normalizer from the independently derived Π_{src}.

What the targeted search did not find. The circulated research Q&A accompanying H-VER077–H-VER080 records a targeted literature search performed on 8 September 2026. It did not find a publication whose stated theorem derives π from a Verlinde hypergroup or from a Weyl-generalised-Chebyshev hypergroup. This is deliberately a narrow search result, not a bibliographic nonexistence theorem.

The project-specific synthesis is the reversed arrow

finite Verlinde/Weyl-generalised-Chebyshev hypergroup → intrinsic half-step refinement → Archimedean cor

12 Why finite algebra alone is insufficient

The entries and structure constants of the finite source lie in algebraic extensions of $\mathbb{Q}$. Any finite sequence of arithmetic operations, matrix products, polynomial elimination, and algebraic extension remains algebraic. The circle constant is transcendental. Therefore the infinite completion above is not cosmetic; it is a logically necessary new kind of edge in the theorem dependency graph.

This separates two roles cleanly:

finite hypergroup	:	selects the exact refinement grammar,
Archimedean completion	:	produces the transcendental readout.

13 Implications and downstream use

The theorem should be imported downstream as an algebraic package, not as a bare decimal constant. A downstream project may rely on the following closed structure:

$$H_{10} \cong \operatorname{Norm} \mathcal{V}_2(A_3^{(1)}),$$
$$\chi_\lambda^{A_3}(\mu) = S_{\lambda\mu}/S_{0\mu},$$
$$C_4 \text{ four-ality}, \qquad T/U \text{ Chebyshev restriction},$$
$$T_2(x) = 2x^2 - 1,$$
$$\Pi_{\mathrm{src}} = \pi,$$
$$\Pi_{\mathrm{src}} = 4\sum_{n\geq 0} \frac{(-1)^n}{2n+1},$$
$$\operatorname{length}(Q_k) = \Pi_{\mathrm{src}}/2,$$
$$\nu(q) = \nu(q^\sharp), \qquad \nu(\overline{q}) = \nu(q^{-1}),$$
$$d\mu(x) = \frac{dx}{\Pi_{\mathrm{src}}\sqrt{1-x^2}}.$$

This matters for later constants work. The present paper now derives the source exponential, its value e_{src} at one, and the source sine and cosine directly from the same completed Chebyshev half-step. A future construction of log and its branch structure may therefore use those objects as parents, but it must still derive logarithmic structure without using a remembered Euler or logarithmic identity circularly.

The primitive cyclotomic-refinement versus prime/Euler-product bridge noted in the circulated Q&A is also still open. It is a separate theorem target, not a consequence of H-VER077.

14 Executable receipts

The generic theorem implementation is

`src/d12rg_hypergroup/archimedean_pi.py`
`src/d12rg_hypergroup/c4_archimedean_boundary.py`

with tests

`tests/test_h_ver077_verlinde_chebyshev_archimedean_pi.py`
`tests/test_h_ver078_source_leibniz_pi.py`
`tests/test_h_ver079_chebyshev_pi_spectral_measure.py`
`tests/test_h_ver080_c4_archimedean_boundary.py`
`tests/test_h_pap001_source_exp_sin_cos.py`
`tests/test_h_ver155_source_analytic_packet.py`

The high-precision Viète routine forms an inscribed/circumscribed polygon interval at 180 decimal digits without reading a library value of π. The Leibniz route is evaluated independently and lands in the same source interval. A library value is used only as a downstream regression witness. The H-PAP001 analytic-tail receipt checks the coefficient recurrence, compact-line $C_{\mathrm{src}} + iS_{\mathrm{src}}$ split, source ODE recurrence, and the Cauchy-product multiplicative law symbolically. It deliberately does not call a pre-existing sine, cosine, or exponential implementation, and it retains a broken-coefficient hostile. This receipt is the finite symbolic certificate reused by the dedicated H-VER155 theorem owner. H-VER155 separately owns the Archimedean completion, differentiability, and endpoint-identification proof boundary.

15 Conclusion

The ten-sector source is already an exact Verlinde, A_3 Weyl-generalised-Chebyshev, and probability-hypergroup object. Its intrinsic order-four direction canonically supplies an ordinary rank-one Chebyshev rail whose multiplication fixes the doubling polynomial. From that point onward the Archimedean proof is rank one: the positive inverse branch is the exact angle bisector, its finite products are polygon perimeter ratios, and the monotone polygon squeeze yields the geometric circle constant. The same source produces the Leibniz expansion and the normalizer of the Chebyshev spectral measure. Its algebraic C_4 transport also proves that one source-derived Π_{src} normalizes all four geometric quadrants and every reciprocal/conjugate branch projection to that boundary.

The same theorem direction now also applies to the elementary analytic functions. Neither sine, cosine, exponential, nor Euler's formula is supplied to the fusion or Chebyshev spectrum. The algebra is constructed first, its half-step and dyadic rotation grammar are proved internally, the Archimedean completion supplies the one-parameter source group, and only then do π, sin, cos, and exp appear as derived readouts.

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