Quarter-Phase Butson–Hadamard Triangular Lift and Euler–Gamma Dual Rails from a Generalized-Chebyshev Hypergroup

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Abstract

This paper develops a source-first analytic continuation of the exact triangular Hadamard packet already owned by the generalized-Chebyshev hypergroup. The finite source contains the pair $T_+(n) = n(n+1)/2$, $T_-(n) = n(n-1)/2$, whose unnormalised Hadamard transform is (n^2, n). The same Hypergroup source also owns the corresponding multiplicative exponential square. We use that source grammar to define a bilateral exponential carrier, derive its local kernel equation, solve the normalised kernel as $\exp(-\exp u)$, and only then recognize the resulting whole-line integral as the conventional Euler Gamma integral. The remaining scalar freedom is fixed by the Hypergroup Product-Law unit: the normalized even zero-row character is one, transported by the paper-local one-shift chart to $\mathcal{G}(1) = 1$. Thus Γ_{Euler} is an endpoint name rather than an upstream premise. Its classical continuation is meromorphic and is retained as the standard recognizable readout, not as the final RH-facing carrier. The derived shift law returns the same two triangular rails and therefore the same (z^2, z) Hadamard pair. Separately, a normalized quaternary Butson-Hadamard matrix $BH(n, 4)$ has total population n^2 and normalized complex excess n, with populations $(T_+(n)-t, t, T_-(n)-t, t)$. Classical Gauss duplication and Euler reflection are then used only downstream of the Gamma reconstruction to expose the complementary quarter rails and the bilateral-entire Euler-Gamma completion, represented by one sine-compensated and one cosine-compensated entire rail. The paper makes no DVJ, zeta, Xi or Riemann-Hypothesis claim. The completed body instead isolates a reusable source-first Gamma bridge and states explicitly which finite, meromorphic, and bilateral-entire objects are proved, recognized, or reserved for downstream use.

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0. Source-first triangular/Gamma equation DAG

Reader contract. The Euler Gamma function is not an upstream premise. It is derived here exclusively from the D12RG Weyl–Chebyshev zorb3 Hypergroup and its certified HG theorem owners; the paper-facing Hypergroup construction is Ref. [2]. The certified path begins with the HG-owned triangular Hadamard pair, uses the already-owned multiplicative exponential rail, normalized Product-Law unit, and source analytic primitives, constructs a bilateral analytic carrier, and only then introduces the conventional name Γ_{Euler}. The meromorphic Euler-Gamma readout is retained because it is the standard recognizable endpoint; it is not, by itself, the final RH-facing carrier in this programme. The downstream sine/cosine compensations retain the whole-line bilateral provenance while giving an entire pair on the full complex plane. DVJ, zeta, Xi, H-PAP002 and H-PAP003 are not premises of this paper. Classical Butson–Hadamard, duplication and reflection formulae are downstream comparison/recognition surfaces.

Imported sources. The following equations are owned by another paper and imported, not redrawn; each item names its owner box. No Gamma premise appears.

- H-PAP001 [1], box front-pap001-archimedean-pi (owner H-VER077): $\Pi_{\mathrm{src}} := \lim_{n\to\infty} P_n^- = \lim_{n\to\infty} P_n^+$ and $\Pi_{\mathrm{src}} = \pi$. Ordinary π enters only through this endpoint recognition.
- H-PAP001 [1], box front-pap001-analytic-packet (owner H-VER155): $\mathrm{Exp}_{\mathrm{src}}(z) := \sum_{k=0}^{\infty} z^k/k!$, $\mathrm{Exp}_{\mathrm{src}}(z+w) = \mathrm{Exp}_{\mathrm{src}}(z)\,\mathrm{Exp}_{\mathrm{src}}(w)$ and $\mathrm{Exp}_{\mathrm{src}}(is) = C_{\mathrm{src}}(s) + iS_{\mathrm{src}}(s)$.
- H-PAP001 [1], box front-pap001-analytic-endpoints (owner H-VER155): $\exp z := \sum_{k=0}^{\infty} z^k/k!$, $\cos s := (\exp(is) + \exp(-is))/2$, $\sin s := (\exp(is) - \exp(-is))/(2i)$, and $\mathrm{Exp}_{\mathrm{src}} = \exp$, $C_{\mathrm{src}} = \cos$, $S_{\mathrm{src}} = \sin$.

Theorem/audit ledger: `H-VER032`, `H-AUD044`.

$$\boxed{\begin{aligned} T_+(n) &= \frac{n(n+1)}{2}, \qquad T_-(n) = \frac{n(n-1)}{2}, \\ H_{\mathrm{ind}} &:= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad H_{\mathrm{ind}} \begin{pmatrix} T_+(n) \\ T_-(n) \end{pmatrix} = \begin{pmatrix} n^2 \\ n \end{pmatrix}. \end{aligned}} \tag{1}$$

Theorem/audit ledger: H-VER032, H-VER033, H-AUD044.

$$\boxed{\begin{gathered}\left(q^{T_+(n)}, q^{T_-(n)}\right) \longmapsto \left(q^{n^2}, q^{n}\right), \qquad q \neq 0, \quad n \in \mathbb{Z}, \\ q^{T_+(n)} q^{T_-(n)} = q^{n^2}, \qquad q^{T_+(n)} / q^{T_-(n)} = q^{n}.\end{gathered}} \tag{2}$$

Theorem/audit ledger: H-VER137, H-VER149, H-AUD073.

$$\boxed{\mathcal{A}(0) = 2, \qquad \mathcal{B}(0) = 0, \qquad h(0) := \frac{\mathcal{A}(0)}{2} = 1.} \tag{3}$$

H-VER149 owns the scalar normalized unit $h(0) = 1$; H-VER137 independently identifies the corresponding supported zero-row object $A_0/2$ as the multiplicative identity. H-PAP004 transports that unit through its paper-local one-shift chart rather than importing $\Gamma_{\mathrm{Euler}}(1) = 1$ as special-function data.

Theorem/audit ledger: H-VER032, H-AUD044.

$$\boxed{\begin{gathered}z \in \mathbb{C}, \qquad T_+(z) = \frac{z(z+1)}{2}, \qquad T_-(z) = \frac{z(z-1)}{2}, \\ H_{\mathrm{ind}} \begin{pmatrix} T_+(z) \\ T_-(z) \end{pmatrix} = \begin{pmatrix} z^2 \\ z \end{pmatrix}.\end{gathered}} \tag{4}$$

Theorem/audit ledger: H-VER033, H-VER155.

$$\boxed{\mathcal{G}(z) := \int_{-\infty}^{\infty} \mathrm{Exp}_{\mathrm{src}}(zu) K(u)\, du, \qquad \mathcal{G}(z+1) = z\mathcal{G}(z).} \tag{5}$$

Theorem/audit ledger: H-VER033, H-VER137, H-VER149, H-VER155, H-AUD073.

$$\boxed{K'(u) + \mathrm{Exp}_{\mathrm{src}}(u) K(u) = 0, \qquad K(u) = \mathrm{Exp}_{\mathrm{src}}(-\mathrm{Exp}_{\mathrm{src}}(u)), \qquad \mathcal{G}(1) = 1.} \tag{6}$$

Theorem/audit ledger: H-VER159, promoted from (5) and (6).

$$\boxed{\begin{gathered}\mathcal{G}_{\mathrm{src}}(z) := \mathcal{G}(z) = \int_{-\infty}^{\infty} \mathrm{Exp}_{\mathrm{src}}(zu - \mathrm{Exp}_{\mathrm{src}}(u))\, du, \qquad \mathrm{Re}\, z > 0, \\ \mathcal{G}_{\mathrm{src}}(z+1) = z\, \mathcal{G}_{\mathrm{src}}(z), \qquad \mathcal{G}_{\mathrm{src}}(1) = 1.\end{gathered}} \tag{7}$$

The H-VER159 source carrier: bilateral and exact on its half-plane, domain-restricted rather than lossy, and not entire. Other papers import it from here.

Theorem/audit ledger: H-VER033, H-VER155.

$$\boxed{\begin{aligned}\mathcal{G}(z) &= \int_{-\infty}^{\infty} \mathrm{Exp}_{\mathrm{src}}(zu - \mathrm{Exp}_{\mathrm{src}}(u))\, du \\ &= \int_0^{\infty} x^{z-1}\exp(-x)\, dx =: \Gamma_{\mathrm{Euler}}(z), \qquad \operatorname{Re} z > 0.\end{aligned}} \tag{8}$$

The symbol Γ_{Euler} is introduced only at this endpoint. The preceding construction does not use a remembered Gamma value, recurrence, product, reflection formula or Mellin normalization.

Theorem/audit ledger: H-VER032, H-VER033, H-AUD044.

$$\boxed{\begin{gathered}\frac{1}{2}\frac{\Gamma_{\mathrm{Euler}}(z+2)}{\Gamma_{\mathrm{Euler}}(z)} = T_+(z), \qquad \frac{1}{2}\frac{\Gamma_{\mathrm{Euler}}(z+1)}{\Gamma_{\mathrm{Euler}}(z-1)} = T_-(z), \\ H_{\mathrm{ind}}\begin{pmatrix} T_+(z) \\ T_-(z)\end{pmatrix} = \begin{pmatrix} z^2 \\ z\end{pmatrix}.\end{gathered}} \tag{9}$$

Theorem/audit ledger: H-VER032, H-VER158, H-AUD044.

$$\boxed{(a,b,c,d) = (T_+(n) - t, t, T_-(n) - t, t)\,,} \tag{10}$$

$$\boxed{\begin{gathered}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i\end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d\end{pmatrix} = \begin{pmatrix} n^2 \\ n \\ n^2 - 4t \\ n\end{pmatrix}, \\ t = \frac{n^2 - (a - b + c - d)}{4}.\end{gathered}}$$

Theorem/audit ledger: H-VER032, H-VER033, H-VER158.

$$\boxed{\begin{gathered}z = \frac{1}{2} + 2w, \qquad P(w) := \Gamma_{\mathrm{Euler}}\left(\frac{1}{4} + w\right)\Gamma_{\mathrm{Euler}}\left(\frac{3}{4} + w\right), \\ 2\frac{P(w+1)}{P(w)} = T_+(z), \qquad 2\frac{P(w+1/2)}{P(w-1/2)} = T_-(z).\end{gathered}} \tag{11}$$

Firewall: Gauss duplication is a downstream classical recognition theorem. It is not used to construct Γ_{Euler}.

Theorem/audit ledger: H-VER077, H-VER155, H-VER158.

$$\boxed{\begin{aligned}E_{\sin}(z) &:= \sin(\pi z)\Gamma_{\mathrm{Euler}}(z) = \frac{\pi}{\Gamma_{\mathrm{Euler}}(1-z)}, \\ E_{\cos}(z) &:= \cos(\pi z)\Gamma_{\mathrm{Euler}}\left(\frac{1}{2} + z\right) = \frac{\pi}{\Gamma_{\mathrm{Euler}}(\frac{1}{2} - z)}.\end{aligned}} \tag{12}$$

Both right-hand sides are entire because $1/\Gamma_{\mathrm{Euler}}$ is entire. Together they are the bilateral-entire Euler-Gamma completion exported by this paper: bilateral because they descend from

the whole-line carrier while retaining the two complementary half-shift/reflection rails, and entire because both components are holomorphic on $\mathbb{C}$. This is not a claim that Γ_{Euler} itself is entire.

Theorem/audit ledger: `H-VER077`, `H-VER155`, `H-VER158`.

$$\boxed{\begin{aligned} E_{\sin}\Big(\frac{1}{4}+w\Big) &= \sin\Big(\pi(\tfrac{1}{4}+w)\Big)\,\Gamma_{\mathrm{Euler}}\Big(\tfrac{1}{4}+w\Big) = \frac{\pi}{\Gamma_{\mathrm{Euler}}(\frac{3}{4}-w)},\\ E_{\cos}\Big(\frac{1}{4}+w\Big) &= \cos\Big(\pi(\tfrac{1}{4}+w)\Big)\,\Gamma_{\mathrm{Euler}}\Big(\tfrac{3}{4}+w\Big) = \frac{\pi}{\Gamma_{\mathrm{Euler}}(\frac{1}{4}-w)}. \end{aligned}} \tag{13}$$

Theorem/audit ledger: `H-VER032`, `H-VER033`, `H-VER077`, `H-VER155`, `H-VER158`, `H-AUD044`.

$$\boxed{\begin{array}{c} (T_+,T_-) \overset{H_{\mathrm{ind}}}{\longleftrightarrow} (z^2,z): \quad T_+(z)+T_-(z)=z^2,\ T_+(z)-T_-(z)=z,\\ \text{HG-derived bilateral carrier} \longrightarrow \underbrace{\Gamma_{\mathrm{Euler}}}_{\text{meromorphic classical readout}} \longrightarrow \begin{pmatrix} E_{\sin}\\ E_{\cos}\end{pmatrix}_{\text{bilateral-entire}},\\ \text{normalized } BH(n,4) \text{ populations retain } (n^2,n),\\ \text{no DVJ, zeta, Xi or RH statement is proved here.} \end{array}} \tag{14}$$

1 Scope and theorem boundary

The declared source is exclusively the D12RG Weyl–Chebyshev zorb3 Hypergroup of Ref. [2]. Repository theorem and audit identifiers are provenance metadata rather than scholarly citations; they are collected in Appendix A and remain visible beside the Section 0 equations solely for proof-DAG audit.

The paper starts from the Hypergroup-owned linear/quadratic spectral pair, its exact inverse-Hadamard triangular sheets, and its multiplicative exponential realization. The target is a source-first analytic carrier whose first classical endpoint is recognized as the conventional Euler Gamma function. The classical Euler Gamma function is therefore not an input datum. Its meromorphic continuation remains in the paper because it is the standard recognizable special-function readout. It is not, by itself, the final RH-facing carrier required by this programme: the downstream interface is the compensated bilateral-entire sine/cosine pair. The finite Hadamard and Butson literature is used only to identify the standard matrix language in which the same linear/quadratic packet appears.

The proof boundary is deliberately narrow. De Vries–Jordan–Jacobi closure, Mellin–zeta completion, the Riemann xi/Xi functions, and the Riemann Hypothesis are downstream or sibling subjects and are not premises of this paper. Likewise, the finite fourth-root alphabet $\{1,i,-1,-i\}$ is not identified with literal values of the Euler Gamma function at quarter arguments.

The input ledger for the present manuscript is:

Input	Role in H-PAP004	Proof status
D12RG Weyl–Chebyshev zorb3 Hypergroup	linear/quadratic spectral pair, triangular reconstruction, multiplicative rail and analytic primitives	source-proved
real Hadamard theory	standard normalization and excess terminology	external comparison
Butson–Hadamard theory	standard fourth-root complex-Hadamard language	external comparison
Euler Gamma literature	endpoint recognition and later functional cross-checks only	downstream

No numerical value of Gamma, no factorial table, and no special-function identity is used to select the source construction. The repository-local owner map implementing this boundary is recorded separately in Appendix A.

2 The HG source already owns the triangular Hadamard packet

Let the Hypergroup spectral coordinate at integer index n be the ordered pair

$$(n^2, n).$$

The adjacent triangular sheets are not independent premises. They are the inverse unnormalised-Hadamard reconstruction

$$\begin{pmatrix} T_+(n) \\ T_-(n) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} n^2 \\ n \end{pmatrix} = \begin{pmatrix} n(n+1)/2 \\ n(n-1)/2 \end{pmatrix}.$$

Because $n^2 \pm n = n(n \pm 1)$ is even for every integer n, the two sheets are integral on the integer source lattice. Applying the forward transform returns

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} T_+(n) \\ T_-(n) \end{pmatrix} = \begin{pmatrix} n^2 \\ n \end{pmatrix}.$$

Thus the quadratic channel is even under $n \mapsto -n$ while the linear channel is odd; the triangular sheets are exchanged by that reflection. This is the finite source relation used throughout the paper.

The same source relation survives exponentiation. For any nonzero oriented-cover variable q,

$$\left(q^{T_+(n)}, q^{T_-(n)}\right) \longmapsto \left(q^{T_+(n)+T_-(n)}, q^{T_+(n)-T_-(n)}\right) = \left(q^{n^2}, q^{n}\right).$$

This multiplicative square is important later: the analytic construction continues the already-owned unit-shift/multiplicative grammar rather than inventing a Gamma recurrence and reading the Hypergroup backwards. The precise repository owners and certificates are listed in Appendix A.

3 Real Hadamard populations recover the triangular pair

Let H be a normalized real Hadamard matrix of order n:

$$H \in \{+1, -1\}^{n\times n}, \qquad HH^T = nI_n,$$

with first row and first column equal to $+1$. Write N_+ and N_- for the total populations of $+1$ and -1 entries. The terminology *excess* for the sum of all entries is standard; Best's paper studies the associated maximum-excess problem [3].

Proposition 3.1 (Normalized real-Hadamard triangular populations)**.** *For a normalized real Hadamard matrix of order n,*

$$N_+ = T_+(n) = \frac{n^2+n}{2}, \qquad N_- = T_-(n) = \frac{n^2-n}{2}.$$

Proof. There are n^2 entries, hence

$$N_+ + N_- = n^2.$$

The first row has sum n. Every other row is orthogonal to the first row, so every other row has sum zero. Consequently the sum of all entries is n, i.e.

$$N_+ - N_- = n.$$

Solving the two equations gives the stated triangular populations. □

The point is simpler than a maximum-excess statement. Normalization and orthogonality alone force the pair

$$(N_+, N_-) = \big(T_+(n), T_-(n)\big),$$

so the unnormalised two-point Hadamard transform returns exactly (n^2, n). No extremal optimization is used.

4 The quaternary Butson–Hadamard lift

Butson's generalized Hadamard construction replaces the real alphabet by roots of unity [4]. In the quaternary case a matrix $H \in BH(n,4)$ has entries in

$$\mu_4 = \{1, i, -1, -i\}$$

and obeys

$$HH^* = nI_n.$$

We use the normalized convention in which the first row and first column are all 1, as in the modern quaternary-Hadamard literature [6].

Let (a,b,c,d) count the entries equal respectively to $1, i, -1, -i$. Then

$$a+b+c+d = n^2.$$

The first row contributes total sum n, while every other row is orthogonal to it. Hence

$$a + ib - c - id = n.$$

Taking real and imaginary parts yields

$$a - c = n, \qquad b - d = 0.$$

Writing $b = d = t$ gives the exact population form

$$\boxed{(a,b,c,d) = (T_+(n) - t,\ t,\ T_-(n) - t,\ t)\,.}$$

Since all populations are nonnegative integers,

$$0 \le t \le T_-(n).$$

This is a necessary range for the parameter attached to any normalized $BH(n,4)$ matrix; the statement does not assert that every integer in this range is realized by a Butson matrix of every order.

Theorem 4.1 (Normalized quaternary triangular decomposition)**.** *Every normalized $BH(n,4)$ population vector has the form above for a unique integer $t = b = d$. Conversely, once the population vector itself is known, t and both triangular populations are recovered exactly.*

The four populations therefore retain more information than the two scalar invariants n^2 and n. Projecting immediately to those two invariants forgets t. Section 5 makes that loss explicit in Fourier coordinates.

The fourth-root alphabet is a finite cyclotomic phase alphabet. Nothing in this section identifies the symbols $1, i, -1, -i$ with $\Gamma_{\text{Euler}}(0)$, $\Gamma_{\text{Euler}}(1/4)$, $\Gamma_{\text{Euler}}(1/2)$ or $\Gamma_{\text{Euler}}(3/4)$; in particular the first of those Gamma values is singular. The analytic quarter rails enter only after Gamma has been independently reconstructed.

5 Four-point Fourier organization and quarter phase

For the ordered population vector

$$N = (a, b, c, d)^T$$

use the unnormalised four-point Fourier matrix

$$F_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}, \qquad F_4^* F_4 = 4I_4.$$

This is the standard Fourier/complex-Hadamard normalization; see, for example, the survey of Tadej and Zyczkowski [5]. The inverse is $F_4^{-1} = \frac{1}{4}F_4^*$.

Substituting the quaternary triangular population vector gives

$$\boxed{F_4 \begin{pmatrix} T_+(n) - t \\ t \\ T_-(n) - t \\ t \end{pmatrix} = \begin{pmatrix} n^2 \\ n \\ n^2 - 4t \\ n \end{pmatrix}.}$$

Thus the zeroth and first Fourier moments are precisely the Hypergroup linear/quadratic pair, up to the displayed order:

$$\widehat{N}(0) = n^2, \qquad \widehat{N}(1) = n.$$

The conjugate third moment equals n as well, while the second moment retains the extra population coordinate:

$$\widehat{N}(2) = n^2 - 4t, \qquad t = \frac{n^2 - \widehat{N}(2)}{4}.$$

Consequently the full F_4 transform is lossless, whereas the projection

$$N \longmapsto (\widehat{N}(0), \widehat{N}(1)) = (n^2, n)$$

forgets the one-dimensional population parameter t. On the affine family above, changing t moves in the direction

$$(-1, 1, -1, 1),$$

which is invisible to those first two moments but visible to $\widehat{N}(2)$. This is the precise finite information-loss statement used later in the paper.

The role of quarter phase here is therefore finite and algebraic:

$$\mu_2 = \{1, -1\} \longrightarrow \mu_4 = \{1, i, -1, -i\}, \qquad H_2 \longrightarrow F_4.$$

The analytic 1/4 and 3/4 Gamma rails developed later are a separate construction and are not assumed at this stage.

6 Complex triangular rails

The finite HG identities do not need a special-function continuation in order to acquire a complex parameter. The triangular sheets themselves are polynomials, so define for every $z \in \mathbb{C}$

$$T_+(z) := \frac{z(z+1)}{2}, \qquad T_-(z) := \frac{z(z-1)}{2}.$$

The same unnormalised Hadamard matrix then gives the polynomial identity

$$\boxed{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} T_+(z) \\ T_-(z) \end{pmatrix} = \begin{pmatrix} z^2 \\ z \end{pmatrix}} \qquad (z \in \mathbb{C}).$$

Conversely,

$$\boxed{\begin{pmatrix} T_+(z) \\ T_-(z) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} z^2 \\ z \end{pmatrix}.}$$

Proposition 6.1 (Complex triangular extension)**.** *The integer triangular packet owned by the HG source has a unique polynomial extension of degree at most two to the complex source coordinate z, and its Hadamard output remains exactly (z^2, z).*

Proof. The displayed formulae are polynomial identities in z. They hold for all nonnegative integers by the finite HG theorem and therefore, or by direct expansion, hold identically on $\mathbb{C}$. No analytic continuation of a factorial or special function occurs at this stage. □

This distinction is important. The present complexification extends the *source polynomials*; the analytic carrier constructed below is a separate operation. Thus no existence or uniqueness theorem for Gamma is being smuggled into the passage from n to z.

7 Bilateral analytic realization of the unit-shift rail

The linear Hadamard output supplies the multiplicative factor z. A single unit-shift carrier therefore contains both triangular sheets if it obeys

$$\boxed{\mathcal{G}(z+1) = z\,\mathcal{G}(z).}$$

Indeed, wherever the quotients are literal,

$$\frac{\mathcal{G}(z+2)}{\mathcal{G}(z)} = z(z+1) = 2T_+(z), \qquad \frac{\mathcal{G}(z+1)}{\mathcal{G}(z-1)} = z(z-1) = 2T_-(z).$$

This is the analytic form of the same two-rail packet, not an imported Gamma recurrence.

We now ask whether that source shift law has a bilateral exponential realization using the already-derived source exponential. Write

$$\mathcal{G}(z) = \int_{-\infty}^{\infty} e^{zu} K(u)\, du.$$

For the forcing argument it is enough to assume that there is a real σ such that K is locally absolutely continuous and

$$e^{\sigma u} K(u), \quad e^{\sigma u} K'(u), \quad e^{(\sigma+1)u} K(u) \in L^1(\mathbb{R}),$$

with

$$\lim_{u \to \pm\infty} e^{(\sigma+it)u} K(u) = 0$$

for every real t on the vertical line used below. These hypotheses are only an admissible transform class; they do not contain a Gamma formula.

For $z = \sigma + it$, integration by parts gives

$$z\mathcal{G}(z) = -\int_{-\infty}^{\infty} e^{zu} K'(u)\, du,$$

whereas the shift law gives

$$\mathcal{G}(z+1) = \int_{-\infty}^{\infty} e^{zu} e^{u} K(u)\, du.$$

Hence

$$\int_{-\infty}^{\infty} e^{zu}\big(K'(u) + e^{u} K(u)\big)\, du = 0$$

for every $t \in \mathbb{R}$. After multiplication by $e^{\sigma u}$ this is the Fourier transform of an L^1 function. Fourier-transform uniqueness therefore forces

$$\boxed{K'(u) + e^{u} K(u) = 0}$$

almost everywhere, and local absolute continuity promotes the identity to the usual differential statement.

Solving this first-order equation gives

$$K(u) = Ce^{-e^{u}}.$$

The remaining scalar is not fixed by importing the remembered value $\Gamma_{\mathrm{Euler}}(1) = 1$. The Hypergroup source already owns its unit. H-VER149 gives the entire Product-Law zero row

$$\mathcal{A}(0) = 2, \qquad \mathcal{B}(0) = 0, \qquad h(0) = \frac{\mathcal{A}(0)}{2} = 1,$$

and H-VER137 independently identifies the corresponding supported $A_0/2$ object as the multiplicative identity. The paper now requires the scalar realization to be unital. In the one-shift carrier chart used here, the source zero/floor unit is represented at the regular carrier argument 1, so unit preservation gives

$$\boxed{\mathcal{G}(1) = 1}.$$

This is the precise sense in which the source floor fixes the normalization: the HG theorem owns the unit $h(0) = 1$, while the coordinate transport $0_{\mathrm{src}} \mapsto 1_{\mathcal{G}}$ is paper-local and explicit. With that source normalization, the elementary substitution $x = e^{u}$ gives

$$\int_{-\infty}^{\infty} e^{u - e^{u}}\, du = \int_{0}^{\infty} e^{-x}\, dx = 1,$$

and therefore fixes $C = 1$.

Theorem 7.1 (Forced bilateral unit-shift kernel)**.** *Within the admissible bilateral-transform class above, the HG unit-shift law and unit normalization determine the kernel uniquely:*

$$\boxed{K(u) = e^{-e^{u}}, \qquad \mathcal{G}(z) = \int_{-\infty}^{\infty} e^{zu - e^{u}}\, du.}$$

The integral converges absolutely for $\operatorname{Re} z > 0$.

Proof. The forcing and normalization arguments above prove uniqueness. For $u \to +\infty$, the factor $e^{-e^{u}}$ dominates every exponential. For $u \to -\infty$, it tends to 1, so the remaining factor behaves as e^{zu} and is integrable precisely when $\operatorname{Re} z > 0$. □

This is the first new analytic completion in H-PAP004. Its input is the HG unit-shift rail plus the already-owned source exponential; Euler Gamma has still not been used.

8 Endpoint recognition as the Euler Gamma function

Define the source-derived carrier by the preceding bilateral integral,

$$\mathcal{G}_{\mathrm{src}}(z) := \int_{-\infty}^{\infty} e^{zu-e^u}\,du, \qquad \operatorname{Re} z > 0.$$

The finite source fixes the shift grammar, the bilateral forcing theorem fixes the kernel, and the source-unit normalization fixes its only scalar freedom. Only now do we compare the resulting object with conventional special-function notation.

The resulting equality on the initial half-plane is exact. Under its classical continuation, however, Γ_{Euler} is meromorphic. H-PAP004 deliberately retains that meromorphic form because it is the standard recognizable special-function readout. This recognition is not a loss of numerical function values on its stated domain; the limitation is carrier typing. The conventional meromorphic scalar, taken alone, does not retain the bilateral-entire interface required by the downstream RH-facing programme. Section 10 therefore constructs the compensated entire pair without deleting this useful intermediate readout.

Make the single coordinate substitution

$$x = e^u, \qquad dx = e^u\,du.$$

Then

$$\boxed{\mathcal{G}_{\mathrm{src}}(z) = \int_0^{\infty} x^{z-1}e^{-x}\,dx, \qquad \operatorname{Re} z > 0.}$$

The right-hand side is Euler's classical integral definition of the Gamma function; see, for example, DLMF [9]. Therefore the conventional name may be introduced at this endpoint:

$$\boxed{\mathcal{G}_{\mathrm{src}}(z) = \Gamma_{\mathrm{Euler}}(z), \qquad \operatorname{Re} z > 0.}$$

Theorem 8.1 (Euler–Gamma endpoint recognition)**.** *The D12RG Weyl–Chebyshev zorb3 Hypergroup source, together with its already-derived exponential primitive and the bilateral admissibility conditions above, produces the normalized carrier* $\mathcal{G}_{\mathrm{src}}$*. Under the conventional Euler-integral definition,*

$$\boxed{\mathcal{G}_{\mathrm{src}} = \Gamma_{\mathrm{Euler}}}$$

on the half-plane $\operatorname{Re} z > 0$*. No value, recurrence, reflection formula, multiplication formula or integral representation of* Γ_{Euler} *is used upstream.*

This is the same source-first recognition pattern used in H-PAP001 for $\Pi_{\mathrm{src}} = \pi$: the source construction is completed first, and the standard mathematical name is attached only after an exact formula match. The DLMF citation is therefore a downstream cross-check and terminology reference, not theorem ancestry.

The recognition also verifies, rather than assumes, the standard functional relation

$$\Gamma_{\mathrm{Euler}}(z+1) = z\Gamma_{\mathrm{Euler}}(z),$$

recorded classically in DLMF [10]. In H-PAP004 that relation has already been obtained from the HG rail before the name Gamma appears.

9 The Gamma return reconstructs the two triangular rails

Once the endpoint has been recognized, the source shift law can be written in conventional notation. Applying it twice gives the meromorphic identities

$$\Gamma_{\mathrm{Euler}}(z+2) = z(z+1)\Gamma_{\mathrm{Euler}}(z),$$

and

$$\Gamma_{\mathrm{Euler}}(z+1) = z(z-1)\Gamma_{\mathrm{Euler}}(z-1).$$

Therefore, on every open set where the displayed quotients are literal,

$$\frac{1}{2}\frac{\Gamma_{\mathrm{Euler}}(z+2)}{\Gamma_{\mathrm{Euler}}(z)} = T_+(z), \qquad \frac{1}{2}\frac{\Gamma_{\mathrm{Euler}}(z+1)}{\Gamma_{\mathrm{Euler}}(z-1)} = T_-(z).$$

The right-hand sides are entire polynomials. Consequently each quotient has the unique removable/meromorphic continuation represented by that polynomial, including across the apparent singular points inherited from individual Gamma factors.

Theorem 9.1 (Gamma recognition/return square)**.** *Let*

$$R_+(z) := \frac{1}{2}\frac{\Gamma_{\mathrm{Euler}}(z+2)}{\Gamma_{\mathrm{Euler}}(z)}, \qquad R_-(z) := \frac{1}{2}\frac{\Gamma_{\mathrm{Euler}}(z+1)}{\Gamma_{\mathrm{Euler}}(z-1)}$$

where the quotients are initially defined, and extend them by the identities above. Then

$$\boxed{R_+(z) = T_+(z), \qquad R_-(z) = T_-(z), \qquad \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} R_+(z) \\ R_-(z) \end{pmatrix} = \begin{pmatrix} z^2 \\ z \end{pmatrix}.}$$

Thus the recognized Euler Gamma carrier returns the complete complex HG triangular packet, not merely its linear recurrence multiplier.

The direction of proof remains important. The return square is an exact recognition receipt. It does not reverse the source ancestry and it is not used to manufacture the original HG pair.

10 Bilateral-entire Euler-Gamma completion: sine and cosine

The derived Gamma carrier is meromorphic after its classical continuation; H-PAP004 does not call Gamma itself entire and does not remove that classical representation. What becomes entire are two compensated dual rails obtained only after endpoint recognition. We call their ordered pair the *bilateral-entire Euler-Gamma completion*: "bilateral" records that it descends from the whole-line $u \in \mathbb{R}$ carrier while keeping the complementary half-shift/reflection rails together, and "entire" means that both compensated components are holomorphic on all of $\mathbb{C}$.

Euler reflection gives

$$\Gamma_{\mathrm{Euler}}(z)\Gamma_{\mathrm{Euler}}(1-z) = \frac{\pi}{\sin(\pi z)}$$

and, after the half-shift $z \mapsto \frac{1}{2} + z$,

$$\Gamma_{\mathrm{Euler}}(\tfrac{1}{2}+z)\Gamma_{\mathrm{Euler}}(\tfrac{1}{2}-z) = \frac{\pi}{\cos(\pi z)};$$

see DLMF [11]. Hence define

$$\boxed{E_{\sin}(z) := \sin(\pi z)\Gamma_{\mathrm{Euler}}(z) = \frac{\pi}{\Gamma_{\mathrm{Euler}}(1-z)},}$$

and

$$\boxed{E_{\cos}(z) := \cos(\pi z)\Gamma_{\mathrm{Euler}}(\tfrac{1}{2}+z) = \frac{\pi}{\Gamma_{\mathrm{Euler}}(\frac{1}{2}-z)}.}$$

The resulting ordered carrier is therefore

$$\boxed{\mathcal{E}_\Gamma^{\mathrm{bil}}(z) := \begin{pmatrix} E_{\sin}(z) \\ E_{\cos}(z) \end{pmatrix} = \begin{pmatrix} \pi/\Gamma_{\mathrm{Euler}}(1-z) \\ \pi/\Gamma_{\mathrm{Euler}}(\frac{1}{2}-z) \end{pmatrix}, \qquad z \in \mathbb{C}.}$$

The reciprocal Gamma function is entire, with zeros at the nonpositive integers; this standard analytic fact is recorded in DLMF [9]. Composition with an affine map therefore proves that both right-hand sides above are entire functions of z.

The two forms are not unrelated copies. They are exact opposing half-shift translates:

$$\boxed{E_{\cos}(z) = E_{\sin}(z + \tfrac{1}{2}).}$$

Indeed, $\sin(\pi(z+\frac{1}{2})) = \cos(\pi z)$ and the Gamma arguments agree. Their zero sets are correspondingly shifted:

$$E_{\sin}(z) = 0 \quad \Longleftrightarrow \quad z = 1, 2, 3, \ldots,$$

while

$$E_{\cos}(z) = 0 \quad \Longleftrightarrow \quad z = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots.$$

At the opposite integer or half-integer poles of the Gamma factors, the trigonometric zero cancels the pole rather than producing a singularity.

Proposition 10.1 (Entire sine/cosine dual pair)**.** *The endpoint-derived Euler Gamma function admits two complementary entire compensations $E_{\sin}$ and $E_{\cos}$ related by a half shift. Their entireness is a downstream consequence of Euler reflection and reciprocal Gamma analyticity; it is not a premise in the HG-to-Gamma reconstruction.*

These two rails are the analytic objects that will be specialized to the 1/4 and 3/4 quarter deck in the next section.

11 The quarter-step $1/4, 1/2, 3/4$ deck

The half-shift relation of the preceding section has a canonical quarter-coordinate refinement. Put

$$s = \frac{1}{2} + 2w$$

and introduce the four Euler–Gamma rails

$$G_{1,+}(w) = \Gamma_{\mathrm{Euler}}(\tfrac{1}{4} + w), \qquad G_{3,+}(w) = \Gamma_{\mathrm{Euler}}(\tfrac{3}{4} + w),$$

$$G_{1,-}(w) = \Gamma_{\mathrm{Euler}}(\tfrac{1}{4} - w), \qquad G_{3,-}(w) = \Gamma_{\mathrm{Euler}}(\tfrac{3}{4} - w).$$

These four functions are not a fourth-root phase alphabet. They are meromorphic analytic rails whose arguments are separated by quarter and half steps. Their usefulness is that duplication and reflection close the square in two independent directions.

Gauss duplication on the two same-sign edges gives

$$\boxed{G_{1,+}(w)G_{3,+}(w) = 2^{1-s}\sqrt{\pi}\,\Gamma_{\mathrm{Euler}}(s),}$$

and

$$\boxed{G_{1,-}(w)G_{3,-}(w) = 2^{s}\sqrt{\pi}\,\Gamma_{\mathrm{Euler}}(1-s).}$$

These are classical endpoint identities after the source-first Gamma recognition of Section 8; they are not premises in that recognition. See DLMF [12].

The cross-sign diagonals are fixed by Euler reflection. With

$$X = \frac{\pi}{4} + \pi w,$$

one obtains

$$\boxed{G_{1,+}(w)G_{3,-}(w) = \frac{\pi}{\sin X},}$$

and

$$\boxed{G_{3,+}(w)G_{1,-}(w) = \frac{\pi}{\cos X}.}$$

The reciprocal diagonal readouts are therefore the sine/cosine pair

$$u(w) := \frac{\pi}{G_{1,+}(w)G_{3,-}(w)} = \sin X, \qquad v(w) := \frac{\pi}{G_{3,+}(w)G_{1,-}(w)} = \cos X.$$

The same unnormalised order-two Hadamard transform used on the triangular rails now gives

$$\boxed{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u(w) \\ v(w) \end{pmatrix} = \sqrt{2} \begin{pmatrix} \cos(\pi w) \\ \sin(\pi w) \end{pmatrix}.}$$

Thus the two cross-sign Gamma diagonals do not merely coexist: their reciprocal readouts close under the same sum/difference operation that already organizes the HG triangular packet.

The same-sign product also reproduces the triangular rails directly. Set

$$P(w) := G_{1,+}(w)G_{3,+}(w).$$

The Gamma recurrence gives

$$\frac{P(w+1)}{P(w)} = \left(w + \frac{1}{4}\right)\left(w + \frac{3}{4}\right) = \frac{s(s+1)}{4},$$

while the opposing half-step quotient gives

$$\frac{P(w+\frac{1}{2})}{P(w-\frac{1}{2})} = \left(w - \frac{1}{4}\right)\left(w + \frac{1}{4}\right) = \frac{s(s-1)}{4}.$$

Consequently

$$\boxed{2\frac{P(w+1)}{P(w)} = T_+(s), \qquad 2\frac{P(w+\frac{1}{2})}{P(w-\frac{1}{2})} = T_-(s),}$$

and a final Hadamard sum/difference returns

$$\boxed{T_+(s) + T_-(s) = s^2, \qquad T_+(s) - T_-(s) = s.}$$

The quarter deck therefore returns the same complexified linear/quadratic packet that was the HG starting datum.

It is essential not to identify the analytic Gamma rails with the finite quarter-phase alphabet

$$\mu_4 = \{1, i, -1, -i\}.$$

For example, the literal quartet $\Gamma(0), \Gamma(1/4), \Gamma(1/2), \Gamma(3/4)$ cannot be that alphabet, because $\Gamma(0)$ is a pole. The relation is organizational rather than an equality of values: the finite Butson/Fourier construction and the analytic Gamma construction both carry a natural four-way quarter indexing, and both preserve distinguished linear/quadratic readouts. Complex Hadamard matrices and their Fourier examples are reviewed in [5]; the Gamma recurrence, reflection and duplication identities are recorded in DLMF [10, 11, 12].

Proposition 11.1 (Quarter-deck return)**.** *After the HG-derived Euler-Gamma endpoint has been recognized, the four quarter-shifted Gamma rails close simultaneously under duplication and reflection. Their same-sign quotient rails recover $T_+(s)$ and $T_-(s)$, while their reciprocal cross-sign diagonals have an exact Hadamard sine/cosine readout. None of these classical endpoint identities is used as a premise in the source-first Gamma reconstruction.*

12 In the literature

The separate ingredients used by H-PAP004 have substantial prior literature. The point of this section is therefore not to relabel standard objects, but to state precisely which conjunction is being used here and which parts are externally conventional.

For real Hadamard matrices, the sum of all entries is the classical "excess" studied by Best [3]. For a normalized real Hadamard matrix the first row has sum n and every other row has sum zero. Hence the total excess is n, while the total number of entries is n^2. Splitting the entries into $+1$ and -1 populations then gives

$$N_+ = \frac{n^2+n}{2} = T_+(n), \qquad N_- = \frac{n^2-n}{2} = T_-(n).$$

This triangular interpretation is an immediate consequence of standard normalization and orthogonality, even though that terminology is not the usual one in the Hadamard literature.

Butson's generalized Hadamard matrices supply the finite root-of-unity extension [4]. Modern complex-Hadamard terminology and the role of Fourier matrices are reviewed by Tadej and Zyczkowski [5]. For the quaternary alphabet $\{1, -1, i, -i\}$, the excess problem has also been studied directly by Kharaghani and Seberry [7]. Classification work on quaternary complex Hadamard matrices includes Lampio–Szollosi–Ostergard [8]; recent work on normalized quaternary Hadamard matrices includes Kharaghani, Tayfeh-Rezaie and Zaitsev [6].

There is independently a literature connection between triangular numbers and Gamma-type interpolation. Abbott and Wu prove a Bohr–Mollerup-style uniqueness theorem for interpolating the triangular numbers [13]. That result is conceptually adjacent to H-PAP004, but it is not used as a source theorem here: the present paper starts from the HG-owned pair (n^2, n), reconstructs the two triangular rails by Hadamard sum/difference, and derives its analytic carrier by the bilateral-kernel argument of Sections 7–8.

The classical facts used only for endpoint recognition and downstream comparison are recorded individually in DLMF: Euler's integral and the meromorphic/reciprocal-entire continuation statement [9], the unit recurrence [10], Euler reflection [11], and Gauss duplication [12].

A targeted literature search accompanying this work located all of those component theories but did not locate the specific package

$$\begin{gathered}\text{quarter phase / Gamma quarter structure}\\ \downarrow\\ \text{complex Hadamard / Fourier organization}\\ \downarrow\\ (T_+, T_-)\\ \downarrow\\ (n^2, n)\end{gathered}$$

stated as one construction. This is a search report, not a universal priority claim. The paper's mathematical claims therefore rest on the explicit derivations given here, not on an assertion that no equivalent formulation can exist elsewhere.

13 Why finite algebra alone is insufficient

The HG and Hadamard identities determine the finite and polynomial rails exactly, but a recurrence by itself cannot identify a unique analytic function. This is the principal boundary between the finite algebra and the analytic theorem of H-PAP004.

Indeed, suppose G is any nonzero solution of

$$G(z+1) = zG(z).$$

If p is a nonzero one-periodic analytic function,

$$p(z+1) = p(z),$$

then

$$\widetilde{G}(z) := p(z)G(z)$$

satisfies the same unit-shift law. Even the normalization $G(1) = 1$ does not remove this ambiguity. For example, for any $\varepsilon \neq 0$,

$$p_\varepsilon(z) = \exp\big(\varepsilon(\cos(2\pi z) - 1)\big)$$

is entire, nonvanishing, one-periodic and obeys $p_\varepsilon(1) = 1$. Thus $p_\varepsilon(z)\Gamma_{\text{Euler}}(z)$ has the same recurrence and normalization as Gamma but is not Gamma.

H-PAP004 therefore does not infer Euler Gamma from the recurrence alone. The additional source-first datum is the admissible bilateral-transform class used in Section 7. Within that class, the shifted transform equation and integration by parts force

$$K'(u) + e^u K(u) = 0.$$

The boundary condition fixes the solution up to a scalar,

$$K(u) = Ce^{-e^u},$$

and the source normalization fixes $C = 1$. Fourier uniqueness on a vertical line then removes the periodic analytic ambiguity above. Only after those steps is the resulting integral recognized as the Euler Gamma integral.

This distinction is parallel to the source-first Archimedean step in H-PAP001: exact finite refinement does not silently grant the infinite completion. The first infinite analytic operation, its domain, and its uniqueness hypotheses must be displayed explicitly.

Proposition 13.1 (Finite-algebra obstruction)**.** *The HG unit-shift recurrence together with $G(1) = 1$ does not uniquely characterize Euler Gamma. Multiplication by a normalized nonzero one-periodic analytic factor preserves both statements. The bilateral transform class and its boundary/uniqueness conditions are therefore load-bearing parts of the H-PAP004 Gamma reconstruction.*

14 Implications and downstream use

H-PAP004 is a bridge paper, but its theorem boundary is deliberately narrow. Its source ancestry ends with the D12RG Weyl–Chebyshev zorb3 Hypergroup, and its analytic endpoint is the independently reconstructed Euler Gamma function together with the quarter-rail receipts proved above. DVJ, zeta, Xi and RH are not premises of any H-PAP004 theorem.

The paper exports four interfaces for downstream use:

1. the complexified triangular pair $T_+(z), T_-(z)$ and its Hadamard image (z^2, z);

2. the source-derived whole-line bilateral carrier, recognized exactly as $\Gamma_{\mathrm{Euler}}(z)$ on its initial half-plane and retained in its conventional meromorphic continuation as a classical readout;

3. the bilateral-entire compensated carrier $\mathcal{E}_\Gamma^{\mathrm{bil}} = (E_{\sin}, E_{\cos})^{\mathsf{T}}$, whose components are entire and related by an exact half shift;

4. the quarter deck whose same-sign and cross-sign products recover the triangular and sine/cosine readouts.

For the RH-facing programme, item 2 is intentionally not the terminal carrier: raw meromorphic Γ_{Euler} does not by itself satisfy the bilateral-entire carrier requirement. It remains essential as the recognizable intermediate readout and as a transparent example of a representation whose conventional scalar typing does not carry all of the source/provenance structure required downstream. The exact equality with the source-derived integral is retained; the limitation is therefore one of carrier typing and provenance, not a claim of lost numerical values.

H-PAP002 may consume the Euler-Gamma endpoint when building its separate DVJ–Mellin–zeta/Xi chain. H-PAP003 may compare its independent quarter-characteristic entire carriers with the quarter organization here. Those are downstream interfaces only: neither sibling paper is required to prove H-PAP004.

The historical scalar named Γ_{DVJ} is also outside this proof ancestry. H-PAP004 uses the explicit name Γ_{Euler} for the special function derived here; similarity of later quarter-step structures does not identify those two database objects.

In particular, no statement in this paper implies RH, a zeta-zero theorem, or a DVJ closure theorem. Its contribution is the narrower mathematical bridge needed so that later work need not import Euler Gamma as an unexplained external premise.

15 Conclusion

What is proved. The starting datum is not Euler Gamma. The D12RG Weyl–Chebyshev zorb3 Hypergroup supplies the linear/quadratic packet (n^2, n), its inverse Hadamard triangular rails, the corresponding multiplicative exponential naturality, and the source unit. In the paper-local shifted coordinate the source floor is transported as

$$0_{\mathrm{src}} \longmapsto 1_{\mathcal{G}}, \qquad h(0) = 1 \longmapsto \mathcal{G}(1) = 1.$$

The single shift law

$$\mathcal{G}(z+1) = z\mathcal{G}(z)$$

then contains both complexified triangular rails. Inside the declared bilateral-transform class, integration by parts and transform uniqueness force

$$K'(u) + e^u K(u) = 0, \qquad K(u) = e^{-e^u},$$

and therefore

$$\mathcal{G}(z) = \int_{-\infty}^{\infty} e^{zu - e^u}\, du = \int_0^{\infty} x^{z-1} e^{-x}\, dx, \qquad \Re z > 0.$$

Only at that endpoint is the source-derived carrier recognized as $\Gamma_{\mathrm{Euler}}(z)$. The conventional Gamma recurrence then returns both triangular rails and hence the same Hadamard pair (z^2, z) with which the construction began.

Why the meromorphic Gamma readout remains visible. The classical continuation of Γ_{Euler} is exact, standard and useful, so it is retained rather than hidden. It is meromorphic, with simple poles at the non-positive integers, while its reciprocal is entire [9]. Thus the scalar Gamma readout is the recognizable intermediate special function but is not, by itself, the final carrier required by the RH-facing D12RG programme. The compensated pair

$$\boxed{\mathcal{E}_{\Gamma}^{\mathrm{bil}}(z) = \begin{pmatrix} E_{\sin}(z) \\ E_{\cos}(z) \end{pmatrix} = \begin{pmatrix} \pi/\Gamma_{\mathrm{Euler}}(1-z) \\ \pi/\Gamma_{\mathrm{Euler}}(\frac{1}{2}-z) \end{pmatrix}}$$

is entire componentwise, carries the opposing half-shift relation $E_{\cos}(z) = E_{\sin}(z + \frac{1}{2})$, and is the bilateral-entire interface exported by this paper. Passing through the conventional meromorphic Gamma coordinate loses no numerical equality; what it does not retain by itself is this two-rail entire typing and the source/provenance structure.

What the quarter-phase calculation adds. The finite quaternary side is independently exact. A normalized $BH(n, 4)$ population has the form

$$N = (T_+(n) - t, t, T_-(n) - t, t)^{\mathsf{T}},$$

and its complete four-point Fourier packet is

$$F_4 N = (n^2, n, n^2 - 4t, n)^{\mathsf{T}}.$$

The first two moments reproduce the distinguished Hypergroup pair but lose t; the full four-point packet recovers it. Downstream of Gamma recognition, the four quarter-shifted Gamma rails close under duplication and reflection, and their same-sign quotient channels return the same $T_{\pm}$ pair. The finite fourth-root alphabet and the analytic Gamma rails therefore share a precise quarter-indexed organization without being identified value-by-value.

Why the analytic class is load-bearing. The recurrence and source-unit normalization alone do not characterize Euler Gamma. Multiplication by any nonzero normalized one-periodic analytic factor preserves both. The bilateral transform class, its boundary conditions, and Fourier uniqueness are therefore part of the proof rather than presentation choices. This is the exact finite-to- analytic boundary of H-PAP004.

What is not proved. Nothing here proves a DVJ closure theorem, a zeta or Xi identity, a zero-free statement, or the Riemann Hypothesis. H-PAP002 and H-PAP003 may consume or compare the interfaces derived here, but neither is an incoming premise. The historical coefficient Γ_{DVJ} is not identified with Γ_{Euler}. The paper also makes no universal priority claim for the combined quarter-phase/Hadamard/triangular/Gamma package: the literature statement in Section 12 is a targeted search report, while the mathematical claims rest on the displayed construction.

Review boundary. The four manuscript-expansion batches now cover the complete body, executable-receipt description, provenance appendix, and external bibliography. Independent mathematical review and final paper disposition remain separate governance steps; they do not alter the theorem ancestry recorded in Section 0.

A Repository provenance ledger

This appendix records repository-local proof provenance. The identifiers below are not used as external scholarly citations in the manuscript body. Section 0 repeats the relevant theorem/audit identifiers beside each equation because the repository Section-0 checker treats them as machine-facing proof metadata.

The public D12RG Hypergroup repository used by this provenance ledger is

`https://gitlab.com/d12rg/d12rg_hypergroup`

Human and machine audits should use that repository root together with the theorem, audit, issue and checkpoint identifiers recorded below.

A.1 Canonical and documentary source owners

Identifier	Repository role
`H-VER032`	canonical triangular-Hadamard/Chebyshev split; the source pair is (n^2, n) and the triangular sheets are reconstructed by inverse Hadamard
`H-VER033`	canonical multiplicative exponential naturality of the triangular packet
`H-VER077`	source-derived Archimedean circle constant used by the analytic primitive packet
`H-VER155`	source-derived exponential, sine and cosine analytic packet
`H-VER149`	canonical entire A/B Product-Law zero row; the normalized even character obeys $h(0) = 1$
`H-VER137`	supported zero-row cross-check; $A_0/2$ is the multiplicative identity
`H-AUD073`	documentary reconciliation of Product-Law zero, the half-shift symmetry floor, and the distinction between source zero labels
`H-AUD044`	documentary archaeology of the integer-power regrading; not a canonical source owner

The paper-facing Hypergroup manuscript is `papers/weyl-verlinde-generalized-chebyshev/weyl-verlinde-generalized-chebyshev.tex`; its scholarly title is cited in the main bibliography as Ref. [2]. The theorem notes above remain the repository's semantic owners.

A.2 Executable receipts and machine-audit entry points

The paper deliberately separates exact executable receipts from analytic proof. The command

```
make paper-h-pap004-verify
```

runs the paper verifier. It checks with exact rational arithmetic:

1. the triangular identities $T_+ + T_- = z^2$ and $T_+ - T_- = z$;
2. the normalized quaternary population formula;
3. the complete four-point Fourier population vector $(n^2, n, n^2 - 4t, n)$ and recovery of t;
4. the rational quarter-coordinate quotient rails returning $T_+(s)$ and $T_-(s)$.

The verifier intentionally does not call a library Gamma implementation. A numerical match to a special-function routine would be downstream evidence, not proof of the source-first analytic identification. The bilateral-kernel uniqueness argument, integration by parts, boundary limits and change of variables are theorem proof obligations in the manuscript.

The strict front-matter contract is checked by

```
make paper-h-pap004-section0-check-strict
```

and the paper itself is built by

```
make paper-h-pap004
```

so the machine-facing Section-0 DAG, exact finite receipts and readable manuscript remain distinct but mutually auditable surfaces.

A.3 H-PAP004 paper-production checkpoints

The relevant paper-production lineage at the start of the four expansion batches is:

Checkpoint	Purpose
`39c2257e`	import received quarter-phase/Hadamard literature note
`6039f301`	preregister H-PAP004 and paper issue
`a809423f`	correct Gamma ownership boundary to HG-only derivation
`69ddbba1`	scaffold the H-PAP001-style manuscript
`b51951db`	record paper-production entry
`63ab3a49`	add Section-0 theorem ledgers and Hypergroup citation
`a3f2ddda`	preregister the four section-expansion batches
`89e8d887`	expand Sections 1–5 and add the provenance appendix
`b1bf0a14`	close Batch 1 after paper and governance checks
`79d6a5fc`	expand Sections 6–10 through the entire dual pair
`2810ec7d`	record Batch 2 provenance and verification receipts
`2de129b5`	close Batch 2 after paper and governance checks
`4eff924c`	expand Sections 11–15 and close the quarter deck
`9a7e5d3e`	record Batch 3 provenance and verification receipts
`d528d37a`	close Batch 3 after paper and governance checks
`903cde8b`	type the source-unit normalization and bilateral-entire Gamma endpoint before Batch 4
`e0011f8a`	record the pre-Batch-4 source-unit and entire-carrier correction checkpoint
`ab39482e`	complete Section 16 and the external-reference audit
`bf8ffd52`	move executable receipts into Appendix A and add the public Hypergroup GitLab audit root

The parent paper issue is `72f/758`. The four manuscript expansion issues are `72f/558` for Sections 1–5, `72f/7a0` for Sections 6–10, `72f/696` for Sections 11–15, and `72f/219` for Section 16 and the final reference pass.

These identifiers provide reproducibility and review navigation only. They must not be promoted into mathematical premises or used in place of the external references cited where standard Hadamard, Butson, Fourier or Gamma facts enter the manuscript.

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