Sine, Cosine, and the Chebyshev Hypergroup
Reciprocal Pairs, Weyl Folding, and Dihedral Holonomy

Luke Kenneth Casson Leighton

24 August 2026

Abstract

At bottom, the construction developed here is sine and cosine. More precisely, it begins with a reciprocal pair q^n, q^{-n} and keeps track of what is retained or lost when their even and odd combinations are formed. On the compact real form these combinations are cosine and sine; on the split real form they are hyperbolic cosine and sine. From this elementary pair arise the first- and second-kind Chebyshev coefficients, a dihedral radial hypergroup, a spherical Hecke algebra, an orbifold compact dual, a torsor lift, holonomic and bispectral characters, a Jordan-three two-jet carrier, prime-log Laplace and wave factorizations, and a positive dihedral holonomy forcing theorem, and a Clifford–dihedral order-24 lift with an explicit source-realization boundary. The purpose of this paper is to expose that common elementary spine while keeping source-derived applications and still-open identifications strictly separated from the generic theorems.

Contents

1 Scope, provenance, and theorem status **2**

2 The reciprocal pair before any quotient **3**

3 At bottom: sine and cosine **4**

4 Weyl radialization and the tangent channel **5**

5 Dihedral parent, hypergroup, torsor, and orbifold **5**

6 Compact and split real forms **5**

7 Holonomic, bispectral, and Jordan-three structure **6**

8 Prime-log scale and the Laplace/wave pair **6**

9 The first-order reciprocal branches **7**

10 Several reflections and positive spectral forcing **8**

11 Triangular Hadamard order-three packet **8**

11.1 Jacobi half-character and finite negative return 9

11.2 Order three and the Paper-5 specialization 10

12 Clifford–dihedral recurrence and the order-24 lift **11**
12.1 Recursive cyclotomic Hadamard packet 12
12.2 Two reflections generate an intrinsic complex structure 12
12.3 Conductor 24 is additional source data 13
12.4 The repository D12 convention 14
12.5 Ancestry and same-parent closure 14

13 Information loss as a recurring structural operation **15**

14 Source boundaries and open problems **16**

15 Application guidance: transport first, collapse last **16**

16 Repository theorem provenance **17**

1 Scope, provenance, and theorem status

The deliberately compressed thesis of this paper is

$$q^{\pm n} \longrightarrow \cos / \sin \longrightarrow T_n/U_{n-1} \longrightarrow \text{radial and oriented geometry.} \tag{1}$$

Most of the apparently disparate terminology used later is generated by asking different questions about this same reciprocal pair: quotienting by reflection, retaining orientation, differentiating the radial coordinate, passing to a compact or split real form, or composing reflections before taking a scalar readout.

The generic theorem chain summarized here is already closed in the repository. In particular, H-PR015 owns the infinite-dihedral radial parent and first-kind Chebyshev hypergroup; H-PR016 owns the orbifold, torsor, Gelfand-pair, holonomic, bispectral, and universal Jordan-three jet descriptions; H-PR017 owns the first-/second-kind even/odd tangent split; H-VER026 owns the prime-local Laplace/wave equation; H-VER027 owns the first-order reciprocal-branch factorization; and H-VER028 owns the positive dihedral holonomy spectral-forcing theorem.

Two important boundaries remain open. H-VER022 still requires a source-owned map from the universal Chebyshev two-jet/Fourier-N carrier to the primitive DVJ realization. H-PR014, the FNIZ programme, also remains open. Downstream Riemann/Xi applications remain separate source-interface questions and are not proved by the generic results in this paper.

The project provenance phrase

> "derived from the Monodromy-free Zorb3 Renormalised RCFT, D12RG"

records the development context. It is not used as a theorem premise. The paper therefore proceeds generic-first: the elementary reciprocal carrier is proved and interpreted before any source-specific specialization is discussed.

No proof of the Riemann hypothesis is claimed here. In particular, prime-local determinant identities are not globalized to $\log \zeta$, and generic reciprocal branches are not identified with Xi-zero, Meyer, Burnol, or primitive DVJ objects without a separately owned intertwiner.

2 The reciprocal pair before any quotient

Let $q \in \mathbb{C}^\times$. The elementary oriented carrier is the ordered reciprocal pair

$$(q^n, q^{-n}), \qquad n \in \mathbb{Z}. \tag{2}$$

The Weyl involution exchanges the two entries,

$$q \longmapsto q^{-1}. \tag{3}$$

Its even and odd combinations are

$$E_n(q) = \frac{q^n + q^{-n}}{2}, \qquad O_n(q) = \frac{q^n - q^{-n}}{2}. \tag{4}$$

Thus E_n is invariant under reciprocal inversion and O_n changes sign. The first recurring information-loss operation in the paper is simply the passage from the ordered pair to E_n alone.

Introduce the Weyl-invariant coordinate

$$x = \frac{q + q^{-1}}{2}. \tag{5}$$

The first-kind Chebyshev polynomial is then

$$\boxed{T_n(x) = \frac{q^n + q^{-n}}{2}.} \tag{6}$$

The normalized odd coefficient is

$$\boxed{U_{n-1}(x) = \frac{q^n - q^{-n}}{q - q^{-1}}.} \tag{7}$$

The denominator fixes the primitive odd normalization. Consequently U_{n-1} is not introduced as an unrelated second polynomial family: it is the normalized anti-invariant coefficient already present in the same reciprocal carrier.

There is an equivalent matrix form. Let $C \in SL(2, \mathbb{C})$ have normalized trace $x = \mathrm{tr}(C)/2$ and let an involution J satisfy

$$JCJ = C^{-1}. \tag{8}$$

Set

$$A = C - xI = \frac{C - C^{-1}}{2}. \tag{9}$$

Then

$$JAJ = -A, \qquad A^2 = (x^2 - 1)I, \tag{10}$$

and the complete matrix power is

$$\boxed{C^n = T_n(x)I + U_{n-1}(x)A, \qquad n \geq 1.} \tag{11}$$

Normalized trace keeps the first term and kills the second. Equation (11) is therefore the finite-dimensional prototype for the distinction between radial data and oriented tangent data used throughout the paper.

3 At bottom: sine and cosine

On the compact reciprocal locus write

$$q = e^{i\theta}, \qquad x = \cos\theta. \tag{12}$$

Then Euler's formula immediately turns (6) into

$$\boxed{T_n(\cos\theta) = \cos(n\theta),} \tag{13}$$

and (7) into

$$\boxed{\sin\theta\, U_{n-1}(\cos\theta) = \sin(n\theta).} \tag{14}$$

Thus the first-/second-kind pair is precisely the normalized cosine/sine pair obtained by retaining both reciprocal orientations.

This elementary observation already explains the multiplication rule that becomes the Chebyshev hypergroup after Weyl radialization. The cosine product formula gives

$$\cos(m\theta)\cos(n\theta) = \frac{1}{2}\cos((m+n)\theta) + \frac{1}{2}\cos((m-n)\theta). \tag{15}$$

After the reflection identification $n \sim -n$, this becomes

$$\boxed{\delta_m * \delta_n = \frac{1}{2}\delta_{m+n} + \frac{1}{2}\delta_{|m-n|}.} \tag{16}$$

The positive coefficients are therefore not inserted by hand: they are the radialized product law of the reciprocal/dihedral parent.

The odd channel obeys the same elementary trigonometric grammar. For example,

$$\cos(m\theta)\sin(n\theta) = \frac{1}{2}\sin((m+n)\theta) + \frac{1}{2}\sin((n-m)\theta), \tag{17}$$

while the odd–odd product returns to the even sector. This is the $\mathbb{Z}_2$-graded structure that sits upstairs of the purely radial first-kind hypergroup.

Differentiation does not introduce a new family either. Exact Chebyshev differentiation gives

$$\boxed{T_n'(x) = nU_{n-1}(x).} \tag{18}$$

The scalar derivative is taken in the folded coordinate x; the matrix tangent axis A in (9) is a separate object. The theorem says that the derivative-derived scalar U_{n-1} is the coefficient multiplying that independently defined odd axis.

One can therefore summarize the elementary core as

$$\boxed{q^{\pm n} \longrightarrow \cos(n\theta),\ \sin(n\theta) \longrightarrow T_n(x),\ U_{n-1}(x).} \tag{19}$$

The remainder of the paper consists largely of keeping careful track of which of these two channels survives each quotient, derivative, real-form change, or holonomy operation.

4 Weyl radialization and the tangent channel

H-PR017 gives an exact meaning to the statement that radialization loses orientation. In the matrix decomposition

$$C^n = T_n(x)I + U_{n-1}(x)A, \tag{20}$$

the T_n term is reflection-even and the $U_{n-1}A$ term is reflection-odd. Normalized trace gives

$$\frac{1}{2}\operatorname{tr}(C^n) = T_n(x), \tag{21}$$

so the odd channel is invisible to the radial scalar character.

Conversely, the pair of coefficients reconstructs the oriented power:

$$C^{\pm n} = T_n(x)I \pm U_{n-1}(x)A. \tag{22}$$

This reconstruction principle is used repeatedly below. Whenever a scalar readout appears to erase a reciprocal modulus or orientation, the first question is whether the discarded odd/operator channel still exists one level upstairs.

5 Dihedral parent, hypergroup, torsor, and orbifold

Let

$$D_\infty = \langle S, F \mid F^2 = 1,\ FSF = S^{-1}\rangle, \qquad K = \{1, F\}, \tag{23}$$

and put $e = (1+F)/2$. The normalized double-coset elements $X_n = eS^n e$ satisfy

$$\boxed{X_m X_n = \frac{1}{2}X_{m+n} + \frac{1}{2}X_{|m-n|}.} \tag{24}$$

Thus the first-kind Chebyshev hypergroup is the exact radial algebra of this infinite-dihedral parent. The pair (D_∞, K) is a discrete Gelfand pair and $e\mathbb{Q}[D_\infty]e$ is its spherical Hecke algebra.

Before reflection chooses a fixed origin, the oriented translation orbit is a $\mathbb{Z}$-torsor. On the compact dual, reciprocal inversion acts on S^1 by $q \mapsto q^{-1}$ and the invariant coordinate is $x = \operatorname{Re} q$. The natural quotient is therefore the orbifold

$$\boxed{[S^1/C_2],} \tag{25}$$

with nontrivial isotropy at $q = \pm 1$, corresponding to $x = \pm 1$. These terms describe distinct levels of the construction: the oriented torsor, the dihedral parent, the radial hypergroup, and its compact orbifold dual are not interchangeable objects.

6 Compact and split real forms

The same complex reciprocal coordinate has two elementary real forms. For

$$q = e^{i\theta}, \tag{26}$$

one obtains the compact pair

$$T_n = \cos(n\theta), \qquad \sin\theta\, U_{n-1} = \sin(n\theta). \tag{27}$$

For

$$q = e^{\chi}, \qquad x = \cosh\chi, \tag{28}$$

one obtains the split pair

$$\boxed{T_n(\cosh\chi) = \cosh(n\chi),} \tag{29}$$

$$\boxed{\sinh\chi\, U_{n-1}(\cosh\chi) = \sinh(n\chi).} \tag{30}$$

The analytic continuation $\theta \mapsto -i\chi$ changes the compact cosine/sine grammar into the split hyperbolic grammar. The theorem does not by itself select a source-specific point such as $x = 3/2$; that selection requires additional source data.

7 Holonomic, bispectral, and Jordan-three structure

The first-kind family satisfies the Chebyshev differential equation

$$\boxed{(1-x^2)T_n'' - xT_n' + n^2T_n = 0,} \tag{31}$$

and the difference equation

$$T_{n+1} - 2xT_n + T_{n-1} = 0. \tag{32}$$

Thus the same family is holonomic/D-finite in x and spectral for a difference operator in n: this is the precise bispectral statement used here.

The universal two-jet

$$(f, f', f'') \tag{33}$$

has truncated derivative shift

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad N^3 = 0, \quad N^2 \neq 0. \tag{34}$$

At a regular point, (31) cuts out a rank-two plane in this three-slot ambient carrier. This proves a universal Jordan-three jet type, not an identification with the primitive DVJ Jordan object. The latter remains the open H-VER022 source-map boundary.

8 Prime-log scale and the Laplace/wave pair

For one arithmetic scale $p > 1$, set

$$\sigma = s\log p, \qquad z = e^{-\sigma} = p^{-s}, \tag{35}$$

and define the reciprocal determinant potential

$$\mathcal{F}(x,\sigma) = -\log(1 - 2xe^{-\sigma} + e^{-2\sigma}). \tag{36}$$

Its convergent expansion is

$$\mathcal{F} = 2\sum_{m\geq 1} \frac{T_m(x)}{m} e^{-m\sigma}. \tag{37}$$

H-VER026 proves

$$\boxed{\left[(1-x^2)\partial_x^2 - x\partial_x + \partial_\sigma^2\right]\mathcal{F} = 0.} \tag{38}$$

On $x = \cos\theta$ this is

$$\boxed{(\partial_\theta^2 + \partial_\sigma^2)\mathcal{F} = 0,} \tag{39}$$

whereas on $x = \cosh\chi$ it is

$$\boxed{(\partial_\sigma^2 - \partial_\chi^2)\mathcal{F} = 0.} \tag{40}$$

The labels "Laplace" and "wave" classify these two real differential forms; no spacetime field equation is asserted. The equation is prime-local. The coefficient $(\log p)^{-2}$ in the ordinary Mellin coordinate depends on p, so the theorem does not globalize naively to $\log\zeta(s)$.

9 The first-order reciprocal branches

Write

$$q = e^\omega \tag{41}$$

and, on a simply connected nonsingular logarithm branch, define

$$F_+ = -\log(1 - e^{-(\sigma-\omega)}), \qquad F_- = -\log(1 - e^{-(\sigma+\omega)}). \tag{42}$$

Then

$$\mathcal{F} = F_+ + F_-, \qquad \mathcal{G} = F_+ - F_-. \tag{43}$$

The primitive equations are merely the two characteristic equations

$$(\partial_\sigma + \partial_\omega)F_+ = 0, \qquad (\partial_\sigma - \partial_\omega)F_- = 0. \tag{44}$$

Equivalently, with

$$\Psi = \begin{pmatrix} \mathcal{F} \\ \mathcal{G} \end{pmatrix}, \qquad J_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{45}$$

one has

$$\boxed{(\partial_\sigma + J_1\partial_\omega)\Psi = 0.} \tag{46}$$

The second-order wave operator follows from the opposite-factor product, not from squaring the same first-order factor:

$$(\partial_\sigma - J_1\partial_\omega)(\partial_\sigma + J_1\partial_\omega) = \partial_\sigma^2 - \partial_\omega^2. \tag{47}$$

The Weyl-odd current is

$$\boxed{J_{\mathrm{odd}} = -\partial_\sigma\mathcal{G} = \partial_\omega\mathcal{F},} \tag{48}$$

and its normalized expansion is

$$\frac{J_{\mathrm{odd}}}{q - q^{-1}} = \sum_{m\geq 1} U_{m-1}(x)e^{-m\sigma}. \tag{49}$$

Thus the second-kind tangent channel is also the current measuring the imbalance between the two reciprocal characteristic branches.

10 Several reflections and positive spectral forcing

The operator-level dihedral mechanism becomes visible before taking a determinant. Let H be an invertible carrier and X an involution with

$$XHX = H^{-1}. \tag{50}$$

Define moving reflections

$$R_a = XH^{-a}. \tag{51}$$

Then

$$R_a R_b = H^{a-b}, \tag{52}$$

and more generally

$$\boxed{R_{a_1} \cdots R_{a_{2r}} = H^k,} \tag{53}$$

with

$$\boxed{k = a_1 - a_2 + a_3 - a_4 + \cdots + a_{2r-1} - a_{2r}.} \tag{54}$$

The relevant dichotomy is $k = 0$ versus $k \neq 0$.

H-VER028 proves the generic positive forcing theorem. If a closed semilinear word has even anti-linear parity, is unitary on a positive finite-dimensional Hilbert space, and source algebra identifies it with H^k for $k \neq 0$, then every eigenvalue h of H satisfies

$$\boxed{|h| = 1.} \tag{55}$$

No determinant is required. In particular, a balanced reciprocal carrier $\mathrm{diag}(h, h^{-1})$ can have determinant one for all h while the full unitary spectrum still forces $|h| = 1$.

11 Triangular Hadamard order-three packet

The same even–odd separation has an exact index-level realization. Define the adjacent triangular exponents

$$T_+(n) = \frac{n(n+1)}{2}, \qquad T_-(n) = \frac{n(n-1)}{2}. \tag{56}$$

Their unnormalised two-point Hadamard transform is

$$\boxed{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} T_+(n) \\ T_-(n) \end{pmatrix} = \begin{pmatrix} n^2 \\ n \end{pmatrix}.} \tag{57}$$

Thus the even output is quadratic in the order index and the odd output is linear.

For nonnegative index these are exactly the two Chebyshev coefficients already encountered in this paper. The first-kind character obeys

$$(1 - x^2)T_n''(x) - xT_n'(x) + n^2 T_n(x) = 0, \tag{58}$$

whereas the tangent derivative is

$$T_n'(x) = nU_{n-1}(x). \tag{59}$$

For negative index the standard extensions $T_{-n} = T_n$ and $U_{-n-1} = -U_{n-1}$ preserve the same even/odd weights. Equation (57) therefore identifies the Hadamard outputs with

the holonomic and tangent weights, respectively. Under $n \mapsto rn$ they scale with degrees two and one:

$$\boxed{(n^2, n) \longmapsto (r^2 n^2, rn).} \tag{60}$$

The Jacobi half-shift makes the parity split more explicit. Completing each triangular exponent gives

$$T_{\pm}(n) + \frac{1}{8} = \frac{1}{2}\left(n \pm \frac{1}{2}\right)^2. \tag{61}$$

Hence

$$\boxed{\left(T_+ + \frac{1}{8}\right) + \left(T_- + \frac{1}{8}\right) = n^2 + \frac{1}{4},} \tag{62}$$

but

$$\boxed{\left(T_+ + \frac{1}{8}\right) - \left(T_- + \frac{1}{8}\right) = n.} \tag{63}$$

The common vacuum correction therefore survives only in the even channel and cancels identically in the oriented channel.

Introduce a logarithmic angular coordinate

$$\ell = n\theta. \tag{64}$$

The quadratic and linear channels have the exponential readouts

$$\boxed{G(\ell) = e^{\ell^2}, \qquad C_{\pm}(\ell) = e^{\pm i\ell}.} \tag{65}$$

The first expression is a Gaussian readout, rather than a character in n; the second is a compact character. A logarithmic half-step gives

$$C_{\pm}(\ell/2)^2 = C_{\pm}(\ell), \qquad G(\ell/2)^4 = G(\ell). \tag{66}$$

Thus halving the logarithmic coordinate halves the linear phase exponent but quarters the quadratic Gaussian exponent. For complex ℓ, principal fourth-root notation requires a branch choice; the displayed fourth-power identity is branch-free. In particular,

$$\sqrt{e^{\ell^2}} = e^{\ell^2/2} \neq e^{(\ell/2)^2} = e^{\ell^2/4}. \tag{67}$$

The two notions of square root must remain distinct.

11.1 Jacobi half-character and finite negative return

The completed upper triangular exponent has the half-centred coordinate

$$\nu = n + \frac{1}{2}, \qquad 2\nu = 2n + 1. \tag{68}$$

Under the affine reflection $r(n) = -n - 1$ one has

$$\boxed{\nu \longmapsto -\nu.} \tag{69}$$

The corresponding heat-kernel realization is the exact exponent identity

$$\boxed{-2\pi x\left(T_+(n) + \frac{1}{8}\right) = -\pi x\left(n + \frac{1}{2}\right)^2.} \tag{70}$$

This decaying heat-kernel realization is distinct from the growing Gaussian readout in Equation (65); both use the same completed quadratic coordinate.

For a positive integer m, quotient the doubled half-centre by $2m$:

$$a_m(n) = 2n + 1 \pmod{2m}. \tag{71}$$

Its image is the complete odd sheet and hence the finite half-character

$$\chi_m(n) = \zeta_{2m}^{2n+1} \tag{72}$$

has negative m-step return,

$$\chi_m(n)^m = -1. \tag{73}$$

Moreover

$$a_m(-n-1) = -a_m(n) \pmod{2m}, \tag{74}$$

so the reflection is exactly reciprocal inversion:

$$\boxed{\chi_m(-n-1) = \chi_m(n)^{-1}.} \tag{75}$$

For odd power radix r, the affine action

$$\sigma_r^{1/2}(n) = rn + \frac{r-1}{2} \pmod{m} \tag{76}$$

intertwines the oriented-cover power map:

$$\boxed{\chi_m\Big(\sigma_r^{1/2}(n)\Big) = \chi_m(n)^r.} \tag{77}$$

Even radices leave the odd sheet and therefore do not restrict to this half-character subpacket.

At $m = 3$ the finite half-character spectrum is

$$\boxed{(q_+, -1, q_-)}, \tag{78}$$

where $q_+ = -\omega^2$ and $q_- = -\omega$ are the conjugate roots of Φ_6. Indeed

$$\boxed{z^3 + 1 = (z+1)\Phi_6(z).} \tag{79}$$

Thus the full negative-return spectrum retains the linear root -1 and the complete nontrivial conjugate factor $\{q_+, q_-\}$ separately. The radix-five power map exchanges q_+ and q_- while fixing -1; this is the retained-root inversion. Only after this branch-sensitive action is complete may the common half-shift transport forget the root sheet.

11.2 Order three and the Paper-5 specialization

For the order-three packet,

$$T_+(3) = 6, \qquad T_-(3) = 3, \tag{80}$$

so

$$\boxed{(6,3) \xrightarrow{H} (9,3).} \tag{81}$$

Paper 5 uses the twelvefold angle

$$\theta_{12} = \frac{\pi}{6}. \tag{82}$$

Therefore

$$\ell = 3\theta_{12} = \frac{\pi}{2}. \tag{83}$$

The two channels become

$$\boxed{e^{\ell^2} = e^{\pi^2/4}}, \tag{84}$$

and

$$\boxed{e^{\pm i\ell} = e^{\pm i\pi/2} = \pm i.} \tag{85}$$

The latter is the quarter-turn specialization of Euler's exponential identity. It is the compact precursor; Euler's formula itself remains the general relation $e^{i\vartheta} = \cos\vartheta + i\sin\vartheta$.

The former coefficient is exactly compatible with the Gaussian factor denoted Γ in the Paper-5 DVJ closure,

$$\Gamma = e^{9\theta_{12}^2} = e^{\pi^2/4}. \tag{86}$$

That closure is written in the square-root coordinate $y = \sqrt{\alpha_{\mathrm{dV}}}$ as

$$\boxed{1 - \Gamma y + y^2 \sum_{m=0}^{\infty} y^{2m}(2\pi)^{-T_m} = 0.} \tag{87}$$

Consequently the quadratic order-three channel 9 supplies the Gaussian coefficient of the source equation whose selected small root is $\sqrt{\alpha_{\mathrm{dV}}}$. In the terminology of that construction this is the DVJ-9 fine-structure-constant channel. Its odd sibling is the order-three Euler quarter-turn channel.

At the logarithmic half-step one obtains simultaneously

$$\boxed{e^{\pm i\ell/2} = e^{\pm i\pi/4}, \qquad e^{(\ell/2)^2} = e^{\pi^2/16} = \Gamma^{1/4}.} \tag{88}$$

This degree-one/degree-two distinction is intrinsic to the Hadamard pair (n^2, n) and does not depend on the order-three specialization.

Remark 11.1. *H-VER033 supplies the source-natural map from the completed triangular half-lattice to the finite negative-return character before the cyclotomic readout. At order three, the complete Φ_6 conjugate factor is evaluated through the H-VER029 polynomial packet while both root sheets remain retained. The unnormalised index Hadamard differs from the unitary presentation only by the common factor $1/\sqrt{2}$; that normalization does not alter the half-character or its naturality.*

12 Clifford–dihedral recurrence and the order-24 lift

The preceding constructions repeatedly combine a reciprocal or oriented coordinate with an involution which reverses it. This is not accidental: the same reflection–translation skeleton already underlies the infinite-dihedral parent of the Chebyshev hypergroup. The purpose of this section is to separate three layers which can otherwise be confused:

1. recursive cyclotomic/Hadamard structure;
2. Clifford closure from anticommuting involutions;
3. finite conductor data selecting an order-24 rotation.

Only the first two are forced by the Hadamard/Clifford algebra itself. The conductor is separate source data.

12.1 Recursive cyclotomic Hadamard packet

The signed cyclotomic pair satisfies

$$\Phi_3(y) = 1 + y + y^2, \qquad \Phi_6(y) = 1 - y + y^2, \tag{89}$$

and therefore

$$\Phi_3(y)\Phi_6(y) = \Phi_3(y^2), \qquad \Phi_6(y^2) = \Phi_{12}(y). \tag{90}$$

Consequently

$$\frac{1}{2}H_0\begin{pmatrix}\Phi_3(y^2)\\ \Phi_6(y^2)\end{pmatrix} = \begin{pmatrix}\Phi_8(y)\\ y^2\end{pmatrix}. \tag{91}$$

At the next dyadic depth,

$$\Phi_{24}(y) = \Phi_6(y^4), \tag{92}$$

so the same splitter gives

$$\frac{1}{2}H_0\begin{pmatrix}\Phi_3(y^4)\\ \Phi_{24}(y)\end{pmatrix} = \begin{pmatrix}\Phi_{16}(y)\\ y^4\end{pmatrix}. \tag{93}$$

H-VER043 freezes the generic form of this recursion. If

$$U_j = y^{2^j} + y^{-2^j}, \tag{94}$$

then the normalized signed packet obeys

$$\frac{1}{2}H_0\mathcal{P}_j(y) = \begin{pmatrix}U_j\\ 1\end{pmatrix}, \qquad \boxed{U_{j+1} = U_j^2 - 2.} \tag{95}$$

Thus the familiar depth-one pair

$$(\Phi_3\Phi_6, \Phi_{12}) \tag{96}$$

is a dyadic descendant of the same signed Hadamard/Chebyshev packet, not a second independent constraint merely because it appears at a deeper level.

The tensor-Hadamard pattern by itself is not yet Clifford structure. H-VER044 concerns commuting involutions and the character table $H_0 \otimes H_0$. Clifford closure instead requires an exact anticommutation law.

12.2 Two reflections generate an intrinsic complex structure

H-VER045 isolates the generic operator theorem. Let S and J be involutions on one retained real module and suppose

$$S^2 = J^2 = I, \qquad JS = -SJ. \tag{97}$$

Define

$$\boxed{K = JS.} \tag{98}$$

Then

$$K^2 = JSJS = -I, \tag{99}$$

so

$$\boxed{K^2 = -I.} \tag{100}$$

No scalar i has been inserted. The square root of minus the identity is constructed from two retained real involutions.

Moreover S, J generate the exact order-eight dihedral operator packet

$$\boxed{\{\pm I, \pm S, \pm J, \pm K\}.} \tag{101}$$

Indeed, with $r = K$ and $s = J$,

$$r^4 = s^2 = I, \qquad srs = r^{-1}. \tag{102}$$

This is simultaneously the rank-two Clifford relation with two +1-square generators and the symmetry algebra of a real square.

On the canonical two-sector index,

$$S_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad J_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{103}$$

so

$$K_0 = J_0 S_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{104}$$

Thus the intrinsic complex structure is literally the real ninety-degree rotation.

There is a retrospective concrete instance already in H-VER035. That theorem owns a sheet reflection X and a half-period complex structure J_{hp} with

$$X^2 = I, \qquad J_{\mathrm{hp}}^2 = -I, \qquad X J_{\mathrm{hp}} = -J_{\mathrm{hp}} X. \tag{105}$$

Putting

$$S_{\mathrm{hp}} = X J_{\mathrm{hp}} \tag{106}$$

gives another involution,

$$S_{\mathrm{hp}}^2 = I, \tag{107}$$

and

$$\boxed{X S_{\mathrm{hp}} = J_{\mathrm{hp}}.} \tag{108}$$

Hence H-VER035 already contained a concrete H-VER045 packet before the generic Clifford–dihedral theorem was isolated.

12.3 Conductor 24 is additional source data

Equation (100) does not select a finite order. Once $K^2 = -I$, every angle θ gives

$$R(\theta) = \cos\theta I + \sin\theta K. \tag{109}$$

Therefore the integer 24 cannot be inferred from the Clifford algebra alone.

H-VER046 adds the separately source-owned conductor-24 step

$$\theta_{24} = \frac{\pi}{12} \tag{110}$$

and defines

$$\boxed{R_{24} = \cos\frac{\pi}{12} I + \sin\frac{\pi}{12} K.} \tag{111}$$

By the algebraic De-Moivre identity,

$$\boxed{R_{24}^{12} = -I, \qquad R_{24}^{24} = I.} \tag{112}$$

If either source reflection reverses K, then it reverses the rotor:

$$\boxed{SR_{24}S = R_{24}^{-1}.} \tag{113}$$

Over the complexified canonical block, K has eigenvalues $-i, +i$. Writing

$$\beta = \zeta_{24} = e^{i\pi/12}, \tag{114}$$

Equation (111) is therefore similar to

$$\boxed{\mathrm{diag}(\beta^{-1}, \beta),} \tag{115}$$

which is exactly the matrix shape used by the order-24 half-shift carrier. The statement is a generic representation theorem: matching this shape does not by itself identify two source parents.

12.4 The repository D12 convention

A nomenclature correction is essential here. This repository fixes

$$D_{12} = \langle r, x \mid r^6 = 1,\ x^2 = 1,\ xrx = r^{-1} \rangle, \tag{116}$$

so the rotation has order six and the group has order twelve.

The faithful group generated by R_{24} and a reversing reflection has rotation order 24 and therefore order 48. Its central half-turn is $R_{24}^{12} = -I$. Quotienting by $\{\pm I\}$ leaves a rotation of order 12, still one level above the repository D_{12} convention.

The repository D_{12} subgroup is obtained from the square rotation:

$$\bar{r} = [R_{24}^2], \qquad \bar{x} = [S]. \tag{117}$$

Since

$$(R_{24}^2)^6 = -I, \tag{118}$$

we have projectively

$$\bar{r}^6 = 1, \qquad \bar{x}^2 = 1, \qquad \bar{x}\bar{r}\bar{x} = \bar{r}^{-1}. \tag{119}$$

This is exactly Equation (116).

No specific binary-dihedral or Pin/Spin group is claimed here. Such a claim requires additional covering-group and reflection-lift data. The closed theorem is the Clifford-generated rotor, the ordinary dihedral reflection law, the central sign, and the stated projective descent.

12.5 Ancestry and same-parent closure

The recurrence of dihedral mathematics across the construction now has a clear provenance hierarchy:

- the Chebyshev hypergroup is the radial shadow of an infinite dihedral parent;
- first-/second-kind Chebyshev coefficients are its reflection-even and reflection-odd coordinates;
- H-VER040 power maps are endomorphisms of that pointed dihedral source before radialization;

- H-VER042 is another reciprocal dihedral representation;
- H-VER035 is a concrete order-eight Clifford–dihedral block;
- H-VER045 abstracts the two-reflection mechanism;
- H-VER046 discretizes its complex structure when conductor 24 is independently owned;
- H-VER047 derives the finite translation/character Weyl–Clifford pair intrinsically from a retained C_2 sheet and its dual sign character;
- H-VER048 applies that theorem to the actual H-VER043 signed packet.

These are not all the same operator. They are related by source parent, representation, finite quotient, radial shadow, or generic abstraction, depending on the row.

H-REV023 is now closed by H-VER048. The actual H-VER043 signed sheet supplies the nontrivial translation and dual sign character, hence two source involutions with

$$H_{\mathrm{src}}^2 = K_{\mathrm{src}}^2 = I, \qquad H_{\mathrm{src}} K_{\mathrm{src}} = -K_{\mathrm{src}} H_{\mathrm{src}}. \tag{120}$$

Independently, dyadic depth two of the same packet gives

$$\Phi_6(q^4) = \Phi_{24}(q), \tag{121}$$

so conductor 24 is not inferred from the Clifford relation. The same sheet Hadamard together with the phase $i = \beta^6$ then gives an explicit intertwiner from the H-VER046 rotor to the exact H-PR013/H-VER042 $\mathrm{diag}(\beta^{-1}, \beta)$ branch. Thus

$$\boxed{\text{actual signed packet} \Longrightarrow \text{source Clifford pair} + \text{owned conductor 24} \Longrightarrow \text{same-parent D24hs carrier.}} \tag{122}$$

No Riemann-zero or physical application is inferred from this finite source closure.

13 Information loss as a recurring structural operation

Several apparently different obstructions in the programme have the same form: a quotient or scalar readout forgets information that remains available in the parent object.

First, the half-step to full-step map $r \mapsto r^2$ loses the sign sheet. Weyl radialization

$$q \longmapsto \frac{q + q^{-1}}{2} \tag{123}$$

then loses the reciprocal sheet. Generically the composite fibre has four algebraic points

$$\{r, -r, r^{-1}, -r^{-1}\}. \tag{124}$$

Second, normalized trace keeps T_n and erases the $U_{n-1}A$ tangent term. Third, a balanced reciprocal determinant

$$\det \mathrm{diag}(h^k, h^{-k}) = 1 \tag{125}$$

forgets the modulus information that the full operator spectrum retains. These are not failures of the parent structure. They are warnings against taking a lossy readout too early.

The unifying methodological rule is therefore simple: when a radial, trace, determinant, or finite quotient appears to erase the datum needed by a later theorem, inspect the oriented or operator-level parent before introducing a new object.

14 Source boundaries and open problems

The generic spine of the paper is closed, but several application maps remain deliberately open.

The universal Chebyshev two-jet/Jordan-three carrier is not yet the source-owned primitive DVJ $N = 3$ object. H-VER022 records the missing DVJ/GKS Fourier-N source map. Likewise the compact Euler points $x = \pm 1/2$ and the split golden point $x = 3/2$ live in the same generic reciprocal grammar, but no source-owned compact-to-golden selector is claimed here.

The prime-log theorem is local to one arithmetic scale and does not supply a global Euler-product differential equation. The ordered reciprocal logarithm branches of H-VER027 are generic function-valued carriers; they are not automatically Xi-zero, Meyer, Burnol, or boundary-jet objects.

Finally, H-VER028 supplies only the generic forcing implication. A Riemann application would still require an actual zero-specific, source-owned positive groupoid/holonomy word whose complete closed word has nonzero reduced dihedral translation and whose carrier contains the appropriate zero-dependent reciprocal eigenvalue. None of those source-interface statements is proved in this paper.

These boundaries are part of the result. The point of reducing the construction to sine and cosine is not to erase source distinctions, but to make the exact generic algebra simple enough that every remaining application-specific arrow can be stated explicitly.

15 Application guidance: transport first, collapse last

The generic theorems are most useful when downstream applications keep complete typed packets until every relation requiring separate channels has acted. This gives the operational rule

$$\boxed{\text{transport first; collapse last.}} \tag{126}$$

A full invertible transform is not itself information loss. For example, retaining both Hadamard channels preserves the original pair. Likewise, normalization can be conservative when the removed scale remains in the source ledger. Loss occurs when a noninjective readout is retained while its omitted channel, orientation, scale, or route provenance is discarded.

The relevant readouts include Weyl radialization, one-channel projection, trace, determinant, quotient, selector evaluation, and normalization with the removed scale forgotten. Their outputs may be exact without being sufficient to reconstruct the parent packet.

H-VER031 adds a second application rule. A mathematical closure cell may contain a nonidentity return map and may be re-applied compatibly H-VER029 scale semigroup. If every edge of a cell has an independently proved scale-naturality square, then the complete return map has the same naturality. In symbols, edge relations

$$f_{j,r}\sigma_{j-1,r} = \sigma_{j,r}f_j \tag{127}$$

imply

$$\boxed{W_r\sigma_{0,r} = \sigma_{0,r}W.} \tag{128}$$

Thus a closed loop in the mathematical object is not a circular proof. The forbidden operation is a self-justifying edge: an application may not insert the desired endpoint relation in order to define an edge which is then used to prove that same endpoint relation.

For practical use, an application should therefore retain source ownership, domain and codomain, orientation, route labels, and scale provenance for each edge. It should stop at the first unsupported arrow. Only after ambient membership, transport, naturality, and the relevant invariant are established should it take a quotient, determinant, selector, or other collapsed readout.

16 Repository theorem provenance

This initial paper checkpoint is intentionally sourced from theorem receipts already present in this repository. The load-bearing internal sources are H-PR015, H-PR016, H-PR017, H-VER020, H-VER021, H-VER026, H-VER027, and H-VER028, together with the O10 source/proof status ledger. External historical and literature references will be audited in a separate bibliography checkpoint rather than reconstructed from memory.