Recursive Torsor Factorisation and Cyclotomic Exact-Order Strata
Pair-First Homomorphisms in the D12RG Hypergroup Programme

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Abstract

This paper isolates a group-theoretic layer upstream of the Chebyshev hypergroup quotient. A basepoint-independent map between torsors forces an induced group homomorphism; equalizer reflection is then equivalent to injectivity of that homomorphism. For cyclic power maps, kernels, images, nonidentity fibres, and exact-order strata are all controlled by greatest common divisors. On C_{24} this gives a recursive subgroup and torsor factorisation of the signed twelve-return, explains the factorisations of $x^{12}-1$ and $x^{12}+1$, and interprets cyclotomic substitution identities as exact-order strata pulled backwards through power homomorphisms. The pair-first dual-Hadamard packet of H-VER037 is then seen to carry the recursive relation $P_{n^2} = P_n \circ P_n$ through its relative and determinant readouts. A previously suppressed C_3 role is recovered both in the kernel/fibre columns and in the transverse power image $P_8(C_{24}) \cong C_3$, complementary to $P_3(C_{24}) \cong C_8$. The same exact-order theorem also recovers the pentic Φ_5/Φ_{10} signed-fibre and squaring pullbacks. The minimal common unshifted clock is then C_{60}, while the quadratic ancestry closes in $C_{120} \cong C_8 \times C_3 \times C_5$ with C_{24} and C_{40} as coordinate faces sharing the C_8 axis. The common C_5 quotient then gives parallel covers $C_{40} \to C_8$ and $C_{120} \to C_{24}$, yielding the pentic lift $x^{60}+1 = \Phi_8\Phi_{24}\Phi_{40}\Phi_{120}$ and the degree refinement $4+8 \to 16+32$. The common clock is then extended by a generic reciprocal rapidity coordinate and inversion, and commuting mirror actions are resolved by basis-independent character projectors before scalarisation. The construction is lifted to the generic dihedral parent before any radial hypergroup quotient. Literature is reviewed for torsors, cyclic subgroup lattices, cyclotomic polynomials, and Chebyshev hypergroups; the paper distinguishes those classical components from the executable synthesis used here. No Riemann hypothesis or application-specific source-role identification is claimed.

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1 Scope, status, and theorem ownership

The purpose of this paper is to expose one common algebraic mechanism behind several already executable results in the D12RG hypergroup (HG) repository. The primary internal owners are H-VER037 and H-VER055–060. The roles are distinct.

H-VER037 retains a pair-first triangular packet before scalar readout. H-VER055 selects the golden split-SL_2 trace from an unsquared relative sixth-return equalizer. H-VER056 records the parity of that equalizer: reciprocal inversion is invisible, while the global half-turn remains visible. H-VER057 proves the generic torsor-homomorphism, equalizer, and recursive subgroup theorems. H-VER058 retains the kernels, torsor fibres, and exact-order cyclotomic strata suppressed by the image-only form of H-VER057.

The paper is deliberately upstream of any application-specific landing. In particular, it does not infer that a Riemann-zero route, an O10 route, or a separately typed affine D_{12} carrier is automatically one of the generic cyclic or dihedral objects below. Those are source-role questions, not consequences of group theory.

The central organizing statement is

$$\boxed{\begin{array}{c}\text{homomorphisms} \longrightarrow \text{kernels/images and torsor fibres} \\ \longrightarrow \text{exact-order strata} \longrightarrow \text{cyclotomic products}\end{array}}$$

The Chebyshev hypergroup appears downstream after radialization or character normalization. Information discarded at that stage cannot be reintroduced by naming a later scalar readout.

2 Pair-first ancestry: H-VER037 and the relative return

For integer n, H-VER037 retains the triangular exponents

$$T_+(n) = \frac{n(n+1)}{2}, \qquad T_-(n) = \frac{n(n-1)}{2},$$

and the ordered reduced packet

$$\widehat{\mathcal{P}}(z,n) = \operatorname{diag}\left(z^{T_+(n)}, z^{T_-(n)}\right), \qquad z \neq 0.$$

The common vacuum rail may be retained independently as

$$\mathcal{P}_{\mathrm{full}}(v,z,n) = vI_2\widehat{\mathcal{P}}(z,n).$$

The triangular sum and difference identities give

$$\det\widehat{\mathcal{P}}(z,n) = z^{n^2}, \qquad \operatorname{Rel}\widehat{\mathcal{P}}(z,n) = z^{n}.$$

The common factor v survives the full determinant and cancels from the ordered relative readout.

Write $P_n(z) = z^n$. Then the apparently scalar relation

$$z^{n^2} = (z^n)^n$$

is the recursive homomorphism identity

$$\boxed{P_{n^2} = P_n \circ P_n.}$$

Thus the H-VER037 Rule and Rock readouts sit on one composed power-map route. They remain separately typed because the determinant and ordered relative readout forget different packet data.

Reflection interchanges the two triangular sheets:

$$\widehat{\mathcal{P}}(z,-n) = X\widehat{\mathcal{P}}(z,n)X, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Consequently the determinant is reflection-even while the relative readout is inverted. This is the pair-first form of the orientation information that later scalar quotients can lose.

The same discipline is visible in H-VER055 and H-VER056. For a reciprocal pair set

$$A_n = q^n + q^{-n}, \qquad B_n = q^n - q^{-n},$$

and define the unsquared relative sixth-return

$$\nu(q) = \frac{B_6(q)}{144B_1(q)}.$$

On the real reciprocal carrier H-VER055 proves

$$\nu = +1 \iff A_1 = 3, \qquad \nu = -1 \iff A_1 = -3.$$

H-VER056 then proves

$$\boxed{\nu(q^{-1}) = \nu(q), \qquad \nu(-q) = -\nu(q).}$$

The target equalizer therefore retains a half-turn bit while remaining blind to reciprocal sheet exchange. This example motivates the generic information-loss theorem of Section 3.

3 Torsor naturality forces a group homomorphism

Let T_G be a left torsor for a group G and let T_H be a left torsor for H. For $x, y \in T_G$ there is a unique displacement

$$\mathrm{disp}_G(y,x) = g \quad \iff \quad y = g \cdot x.$$

No preferred origin is required. The diagonal relation is canonical:

$$\mathrm{disp}_G(y, x) = e_G \iff y = x.$$

Let $F : T_G \to T_H$. Suppose that for each $g \in G$ the target displacement

$$\mathrm{disp}_H(F(g \cdot x), F(x))$$

is independent of the chosen basepoint x. Define

$$\phi_F(g) = \mathrm{disp}_H(F(g \cdot x), F(x)).$$

Theorem 3.1 (H-VER057 torsor homomorphism)**.** *Basepoint-independent relative transport forces*

$$\boxed{\phi_F(g_1 g_2) = \phi_F(g_1)\phi_F(g_2).}$$

Hence $\phi_F : G \to H$ is a group homomorphism.

Proof. Start the second leg at $g_2 \cdot x$. Basepoint independence gives

$$F(g_1 g_2 \cdot x) = \phi_F(g_1) \cdot F(g_2 \cdot x) = \phi_F(g_1)\phi_F(g_2) \cdot F(x).$$

Uniqueness of torsor displacement gives the result. □

The homomorphism is therefore derived from natural relative transport; it is not an extra hypothesis.

Theorem 3.2 (Equalizer reflection)**.** *The torsor map F reflects the diagonal exactly when ϕ_F is injective:*

$$\boxed{F^{-1}(\Delta_H) = \Delta_G \iff \ker \phi_F = \{e_G\} \iff \phi_F \text{ is injective.}}$$

This theorem clarifies the phrase "equalizer-reflecting". Naturality alone is insufficient: the constant homomorphism sends every relative return to the target identity but reflects no nontrivial equalizer.

Composition is recursive. If $K : T_H \to T_L$ is another natural torsor map, then

$$\boxed{\phi_{K \circ F} = \phi_K \circ \phi_F.}$$

Images are therefore subgroups. For arbitrary endomorphisms, however, composition alone does not imply a nested chain of images. Nesting needs preservation of the previous image. The cyclic power maps below satisfy that extra condition automatically.

Finally, let a quotient $\pi : G \to Q$ kill a nontrivial central half-turn ϵ, while a target map detects it:

$$\pi(\epsilon) = e_Q, \qquad \phi(\epsilon) \neq e_H.$$

Then ϕ cannot factor through π. Otherwise

$$\phi(\epsilon) = \bar{\phi}(\pi(\epsilon)) = \bar{\phi}(e_Q) = e_H,$$

a contradiction. A quotient that forgets orientation cannot support a downstream map whose target still detects that orientation.

4 Cyclic power maps and the divisor lattice

Write $C_N = \mathbb{Z}/N\mathbb{Z}$ additively and define

$$P_m(k) = mk \pmod{N}.$$

Put $d = \gcd(N, m)$. Then

$$|\ker P_m| = d, \qquad \operatorname{Im} P_m = \langle d \rangle, \qquad |\operatorname{Im} P_m| = \frac{N}{d}.$$

The first isomorphism theorem gives

$$\boxed{C_N/\ker P_m \cong \operatorname{Im} P_m.}$$

For the multiplicative character ζ_N^m the exact order is

$$\boxed{\operatorname{ord}(\zeta_N^m) = \frac{N}{\gcd(N, m)} = |\operatorname{Im} P_m|.}$$

Thus the integer used as a cyclotomic conductor/order is simultaneously the order of an image subgroup, while the complementary gcd is the kernel size.

The subgroup lattice of a finite cyclic group is the divisor lattice: for every divisor $r \mid N$ there is exactly one subgroup of order r. For $N = 24$ the orders are

$$1, 2, 3, 4, 6, 8, 12, 24.$$

In particular, C_3 is not an optional embellishment of the C_{24} lattice. It is one of its canonical divisor-lattice nodes.

Power maps compose:

$$\boxed{P_a \circ P_b = P_{ab}.}$$

Writing

$$d_b = \gcd(N, b), \qquad N_b = \frac{N}{d_b},$$

the second stage acts on a cyclic image of order N_b. Its new kernel has order $\gcd(N_b, a)$, and

$$\boxed{\gcd(N, ab) = \gcd(N, b)\gcd\left(\frac{N}{\gcd(N, b)}, a\right).}$$

This is the recursive kernel form of the same gcd arithmetic that classifies the final conductor.

5 Homomorphism fibres and cyclotomic exact-order strata

Let $\phi : G \to H$ be a group homomorphism. If $h \in \operatorname{Im}\phi$ and $x \in \phi^{-1}(h)$, then

$$\boxed{\phi^{-1}(h) = x \ker \phi.}$$

Hence every nonempty fibre is a torsor for the kernel. The identity fibre is the kernel subgroup; a nonidentity fibre is generally a coset, not a subgroup.

For a root q of exact order r, the power q^m has exact order

$$\boxed{\operatorname{ord}(q^m) = \frac{r}{\gcd(r, m)}.}$$

Therefore a primitive exact-order r stratum lies over the primitive exact-order n stratum precisely when

$$\frac{r}{\gcd(r,m)} = n.$$

Since such an r divides nm, the complete pullback is finite.

The roots of the cyclotomic polynomial Φ_n are exactly the primitive roots of order n. Consequently H-VER058 obtains the monic factorisation

$$\boxed{\Phi_n(x^m) = \prod_{\substack{r \mid nm \\ r/\gcd(r,m)=n}} \Phi_r(x).}$$

This formula is best read as a pullback of exact-order strata through the power homomorphism P_m.

For $m = 2$ one obtains

$$\begin{aligned}
\Phi_1(x^2) &= \Phi_1(x)\Phi_2(x), \\
\Phi_2(x^2) &= \Phi_4(x), \\
\Phi_3(x^2) &= \Phi_3(x)\Phi_6(x), \\
\Phi_6(x^2) &= \Phi_{12}(x).
\end{aligned}$$

The third identity says that a primitive third root downstairs has preimages of exact orders three and six. The fourth says that a primitive sixth root downstairs has preimages of exact order twelve only. The polynomial factorisations are therefore exact-order shadows of the group homomorphism.

The same theorem retains the pentic companion which is already present in the H-VER039 reciprocal hypergroup packet:

$$\boxed{\Phi_5(x^2) = \Phi_5(x)\Phi_{10}(x), \qquad \Phi_{10}(x^2) = \Phi_{20}(x), \qquad \Phi_{20}(x^2) = \Phi_{40}(x).}$$

The first identity pulls a primitive fifth-order stratum back to exact orders five and ten. The second pulls a primitive tenth-order stratum back to exact order twenty. The third continues the same quadratic ancestry from exact order twenty to exact order forty.

There is also a signed-fibre presentation independent of C_{24}. On μ_{10}, the map $q \mapsto q^5$ has

$$P_5^{-1}(+1) = \mu_5, \qquad P_5^{-1}(-1) = \zeta_{10}\mu_5.$$

The subgroup fibre has exact orders $1, 5$, while the nonidentity torsor fibre has exact orders $2, 10$. Hence

$$\boxed{x^5 - 1 = \Phi_1\Phi_5, \qquad x^5 + 1 = \Phi_2\Phi_{10}.}$$

Thus Φ_5 and Φ_{10} are not inserted into the C_{24} strata. They are the pentic realization of the same generic fibre and exact-order pullback theorem.

Recursive composition now has a polynomial counterpart. Pulling an exact-order stratum backwards through P_b and then P_a gives the same final root set as pulling it back through P_{ab} directly. Different factorisations can expose different intermediate subgroup and torsor routes while preserving one composite return.

6 The C24 twelve-return and the recovered C3 role

Take

$$\mu_{24} = \langle \zeta_{24} \rangle$$

and the signed return map

$$\pi(q) = q^{12} : \mu_{24} \longrightarrow \{\pm 1\}.$$

Its kernel is μ_{12}. The two fibres are therefore

$$\boxed{\pi^{-1}(+1) = \mu_{12}, \qquad \pi^{-1}(-1) = \zeta_{24}\mu_{12}.}$$

The first is a subgroup; the second is the other μ_{12} torsor fibre.

The exact-order strata are

q^{12}	group-theoretic object	exact orders
$+1$	μ_{12} subgroup	$1, 2, 3, 4, 6, 12$
-1	$\zeta_{24}\mu_{12}$ torsor	$8, 24$

and hence

$$x^{12} - 1 = \Phi_1\Phi_2\Phi_3\Phi_4\Phi_6\Phi_{12},$$

while

$$\boxed{x^{12} + 1 = \Phi_8\Phi_{24}.}$$

The negative-return roots do not form a subgroup. Their product is the exact-order decomposition of one nonidentity fibre.

H-PR007 already owns the source CRT product

$$C_{24} \cong C_3 \times C_8.$$

H-VER059 exposes this directly through complementary power images:

$$\boxed{\ker P_3 = P_8(C_{24}) \cong C_3, \qquad \ker P_8 = P_3(C_{24}) \cong C_8.}$$

The two image subgroups have trivial intersection and together generate C_{24}. Thus $P_8(C_{24})$ is a source-transverse C_3 image which is available before any factorisation of the twelve-return is chosen.

H-VER057 originally displayed only the intermediate and final image subgroups for the four ordered two-stage factorisations of P_{12}. The first isomorphism theorem shows that this suppresses half of the structure. H-VER058 restores both kernel columns, while H-VER059 also retains the transverse source C_3 image:

$P_a \circ P_b$	$\ker P_b$	$P_b(C_{24})$	$\ker(P_a\vert_{P_b(C_{24})})$	$P_{12}(C_{24})$	$P_8(C_{24})$
$P_2 \circ P_6$	C_6	C_4	C_2	C_2	C_3
$P_3 \circ P_4$	C_4	C_6	C_3	C_2	C_3
$P_4 \circ P_3$	C_3	C_8	C_4	C_2	C_3
$P_6 \circ P_2$	C_2	C_{12}	C_6	C_2	C_3

This answers the C_3 question in both senses. It appears inside the route-specific kernel/fibre data, and it is also present transversely in every row through the independent source image $P_8(C_{24})$. The last column does *not* assert that P_8 applied after each intermediate map still has image C_3; that stronger claim is false in general.

Every row satisfies

$$|\ker P_b|\,|P_b(C_{24})| = 24$$

and

$$|\ker(P_a|_{P_b(C_{24})})|\,|C_2| = |P_b(C_{24})|.$$

Thus the triadic node is forced by the same first-isomorphism-theorem bookkeeping that produces the final C_2 return, while the transverse P_8 image records the source C_3 axis independently of route choice.

The H-VER037 triadic packet gives another exact appearance. At $n = 3$, Rock is P_3, so on C_{24}

$$|\ker P_3| = 3, \qquad |\operatorname{Im} P_3| = 8.$$

Rule is $P_9 = P_3 \circ P_3$. The second P_3 is injective on C_8 because $\gcd(8, 3) = 1$, so Rule retains the same C_3 kernel and C_8 image. The triadic subgroup is visible before the pair-first packet is collapsed to scalar-only semantics.

7 Common source closure at C60 and lifted closure at C120

The triadic/quartic and pentic families admit a single finite cyclic parent without identifying their separately typed source roles. At the unshifted level,

$$\gcd(12, 10) = 2, \qquad \operatorname{lcm}(12, 10) = 60,$$

so the unique subgroups $C_{12}, C_{10} \subset C_{60}$ satisfy

$$\boxed{C_{12} \cap C_{10} \cong C_2, \qquad \langle C_{12}, C_{10} \rangle = C_{60}.}$$

Thus C_{60} is the minimal common unshifted clock.

There is an important cardinality firewall. Although

$$|C_{12} \times C_{10}| = 120,$$

the direct product is not cyclic: its largest possible element order is only $\operatorname{lcm}(12, 10) = 60$. Standard invariant factors give

$$C_{12} \times C_{10} \cong C_2 \times C_{60},$$

not C_{120}. The cyclic order-120 object appears instead as the next dyadic cover.

Indeed the squaring endomorphism on C_{120} has

$$\boxed{\ker P_2 \cong C_2, \qquad P_2(C_{120}) \cong C_{60},}$$

and therefore

$$\boxed{C_{120}/C_2 \cong C_{60}.}$$

This is the exact cyclic double-cover relation between the lifted and unshifted common clocks.

The closure ladder is

stage	triadic	pentic	intersection	minimal parent
unshifted	C_{12}	C_{10}	C_2	C_{60}
first lift	C_{24}	C_{20}	C_4	C_{120}
full pentic lift	C_{24}	C_{40}	C_8	C_{120}

so the shared dyadic rail grows

$$C_2 \longrightarrow C_4 \longrightarrow C_8,$$

while the common parent stabilizes after the first lift:

$$\boxed{C_{60} \longrightarrow C_{120} \longrightarrow C_{120}.}$$

In particular,

$$\operatorname{lcm}(24, 20) = \operatorname{lcm}(24, 40) = 120.$$

Thus the further pentic $20 \to 40$ lift requires no larger clock.

The prime factorization makes the fully lifted structure transparent:

$$\boxed{C_{120} \cong C_8 \times C_3 \times C_5.}$$

The three axes occur as power images

$$P_{15}(C_{120}) \cong C_8, \qquad P_{40}(C_{120}) \cong C_3, \qquad P_{24}(C_{120}) \cong C_5.$$

They intersect pairwise only in the identity and generate the complete parent.

The triadic and pentic lifted families are then literal coordinate faces:

$$\boxed{C_{24} \cong C_8 \times C_3, \qquad C_{40} \cong C_8 \times C_5.}$$

Consequently

$$\boxed{C_{24} \cap C_{40} \cong C_8, \qquad \langle C_{24}, C_{40} \rangle = C_{120}.}$$

This identifies the common dyadic axis without collapsing the C_3 and C_5 coordinates.

The unshifted parent has the corresponding decomposition

$$C_{60} \cong C_4 \times C_3 \times C_5.$$

Here $C_{12} \cong C_4 \times C_3$ and $C_{20} \cong C_4 \times C_5$, while the original $C_{10} \cong C_2 \times C_5$ occupies the unique C_2 subaxis of the common C_4 direction.

The two quadratic exact-order ladders now sit in one parent:

$$\boxed{24 \longrightarrow 12 \longrightarrow 6 \longrightarrow 3,}$$

and

$$\boxed{40 \longrightarrow 20 \longrightarrow 10 \longrightarrow 5.}$$

Every arrow is the exact-order effect of squaring. Pulling the strata backwards gives the parallel table

triadic/quartic	pentic
$\Phi_3(x^2) = \Phi_3(x)\Phi_6(x)$	$\Phi_5(x^2) = \Phi_5(x)\Phi_{10}(x)$
$\Phi_6(x^2) = \Phi_{12}(x)$	$\Phi_{10}(x^2) = \Phi_{20}(x)$
$\Phi_{12}(x^2) = \Phi_{24}(x)$	$\Phi_{20}(x^2) = \Phi_{40}(x)$

The two columns are therefore different exact-order faces of one C_{120} subgroup lattice rather than unrelated polynomial identities.

This statement is upstream of the dihedral and radial hypergroup layers. A dihedral extension of the C_{120} clock has twice the cardinality; it is not denoted D_{120} when the repository convention uses the subscript for total dihedral order.

8 Pentic cover of the negative-return conductor sectors

The common C_{120} parent carries a second structure which is not visible from the CRT-face statement alone. The fifth-power map removes exactly the C_5 coordinate from both the pentic face and the complete parent:

$$\boxed{P_5 : C_{40} \twoheadrightarrow C_8, \qquad P_5 : C_{120} \twoheadrightarrow C_{24},}$$

with the same kernel C_5 in both cases. Equivalently,

$$\boxed{C_{40}/C_5 \cong C_8, \qquad C_{120}/C_5 \cong C_{24}.}$$

Thus C_{40} and C_{120} are parallel pentic covers of the two negative-return conductor sectors already present in the C_{24} geometry.

The full divisor ladders make the quotient structure explicit:

m	$\ker(P_m\|_{C_{40}})$	$P_m(C_{40})$
5	C_5	C_8
10	C_{10}	C_4
20	C_{20}	C_2
40	C_{40}	C_1

m	$\ker(P_m\|_{C_{120}})$	$P_m(C_{120})$
5	C_5	C_{24}
10	C_{10}	C_{12}
20	C_{20}	C_6
40	C_{40}	C_3

Each row is an instance of the first isomorphism theorem. The first rows are the load-bearing comparison: the same C_5 kernel produces C_8 downstairs from C_{40} and C_{24} downstairs from C_{120}.

The corresponding exact-order pullbacks are

$$\boxed{\Phi_8(x^5) = \Phi_8(x)\Phi_{40}(x),}$$

and

$$\boxed{\Phi_{24}(x^5) = \Phi_{24}(x)\Phi_{120}(x).}$$

They follow directly from the H-VER058 exact-order pullback theorem: primitive eighth roots have fifth-power preimages of exact orders 8 and 40, while primitive twenty-fourth roots have preimages of exact orders 24 and 120.

The signed twelve-return identity therefore has a canonical pentic pullback. Starting from

$$x^{12} + 1 = \Phi_8(x)\Phi_{24}(x)$$

and substituting x^5 gives

$$\begin{aligned} x^{60} + 1 &= \Phi_8(x^5)\Phi_{24}(x^5) \\ &= \boxed{\Phi_8(x)\Phi_{24}(x)\Phi_{40}(x)\Phi_{120}(x)}. \end{aligned}$$

The same factorization follows independently from the negative fibre of

$$P_{60} : C_{120} \longrightarrow \{+1, -1\}.$$

The nonidentity fibre is a torsor for the order-sixty kernel, and its exact-order strata are precisely

$$\boxed{8, 24, 40, 120.}$$

Thus substitution and fibre classification give independent receipts for the same four-factor decomposition.

This also refines the H-PR007 negative-return rank split. The old strata have degrees

$$\deg \Phi_8 = 4, \qquad \deg \Phi_{24} = 8,$$

whereas their nontrivial pentic preimages have

$$\deg \Phi_{40} = 16, \qquad \deg \Phi_{120} = 32.$$

Hence

$$\boxed{(4,8) \longmapsto (16,32) = 4(4,8),}$$

with $4 = \varphi(5)$. The ratio is preserved:

$$\boxed{4 : 8 = 16 : 32 = 1 : 2,}$$

and the complete negative fibre has degree

$$\boxed{4 + 8 + 16 + 32 = 60.}$$

The CRT-face and quotient descriptions are complementary. H-VER060 writes

$$C_{40} \cong C_8 \times C_5, \qquad C_{120} \cong C_8 \times C_3 \times C_5,$$

while the present theorem quotients the same C_5 coordinate:

$$\boxed{(C_8 \times C_5)/C_5 \cong C_8, \qquad (C_8 \times C_3 \times C_5)/C_5 \cong C_8 \times C_3.}$$

This is the exact sense in which C_{120} is the common pentic cover of the two negative-return conductor sectors 8 and 24.

The statement remains generic. It does not identify any downstream application character with a C_{120} coordinate, nor does it infer an application source-role intertwiner from the finite quotient diagram.

9 Reciprocal carrier and commuting-mirror character cube

The common C_{120} parent also admits a generic reciprocal extension which is useful before any application-specific representation is chosen. Let

$$A_{120} = C_{120} \times (\mathbb{R}, +)$$

and let an external involution X act by

$$X : (k, t) \longmapsto (-k, -t).$$

The resulting generalized-dihedral carrier is

$$\mathcal{G}_{120} = A_{120} \rtimes C_2^X.$$

This is an abstract group-theoretic carrier. No identification with an application source or with a particular matrix representation is implied.

9.1 Three order-two roles

The internal half-turn is

$$h = 60 \in C_{120}, \qquad h^2 = e.$$

It must be distinguished from the return-sign quotient

$$P_{60} : C_{120} \longrightarrow C_2^{\mathrm{ret}}, \qquad k \longmapsto (-1)^k,$$

and from the external inversion group C_2^X. The exact sequence

$$1 \longrightarrow C_{60} \longrightarrow C_{120} \xrightarrow{P_{60}} C_2^{\mathrm{ret}} \longrightarrow 1$$

is nonsplit: the unique order-two element of C_{120} is h, but $P_{60}(h) = +1$.

Since $h^{-1} = h$, inversion fixes the half-turn. Therefore

$$XhX = h, \qquad [h, X] = e,$$

and the two involutions generate the canonical Klein four

$$\boxed{\langle h, X\rangle \cong C_2 \times C_2.}$$

The negative sixty-return fibre of H-VER061 is invariant under both operations, so its four exact-order strata

$$8, 24, 40, 120$$

form a finite packet stable under this Klein-four action.

9.2 Finite torsion cannot carry rapidity

Choose a primitive generator $g_8 = 15$ of the unique C_8 subgroup of C_{120}. Then

$$g_8^4 = h.$$

For a lifted element

$$\widetilde{g}_8(t) = (g_8, t)$$

in the connected product,

$$\widetilde{g}_8(t)^4 = (h, 4t).$$

Hence

$$\boxed{\widetilde{g}_8(t)^4 = (h, 0) \quad \Longleftrightarrow \quad t = 0.}$$

This is a special case of a generic torsion theorem. If F is finite and

$$\varphi : F \longrightarrow C_{120} \times (\mathbb{R}, +)$$

is a homomorphism, then for an element g of order n, writing $\varphi(g) = (c, t)$ gives $nt = 0$. Since $(\mathbb{R}, +)$ is torsion-free, $t = 0$. Thus the radial projection of every finite subgroup image is trivial. Combined with H-VER057, this applies after a source-natural torsor transport has actually been proved to induce such a homomorphism.

9.3 Basis-independent mirror projectors

H-VER044 treated two commuting involutions in an explicit Hadamard-square basis. H-VER063 gives the invariant formulation. Let

$$J_1, \ldots, J_r$$

be commuting involutions on a module over a coefficient ring in which 2 is invertible. For

$$\chi = (\chi_1, \ldots, \chi_r), \qquad \chi_j \in \{+1, -1\},$$

define

$$\boxed{\Pi_\chi = 2^{-r} \prod_{j=1}^{r} (I + \chi_j J_j).}$$

Then

$$\Pi_\chi^2 = \Pi_\chi, \qquad \Pi_\chi \Pi_\psi = 0 \quad (\chi \neq \psi), \qquad \sum_\chi \Pi_\chi = I,$$

and

$$J_j \Pi_\chi = \chi_j \Pi_\chi.$$

These projectors, rather than a chosen transform matrix, are the canonical objects.

In a regular orbit basis the corresponding character table is the Walsh–Hadamard transform

$$W_r = H_0^{\otimes r}, \qquad H_0 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

with

$$W_r^2 = 2^r I.$$

Thus the recurring "Hadamard within Hadamard" pattern is precisely the character decomposition of an elementary abelian mirror group C_2^r. The cyclic DFT remains separately typed: it resolves cyclic phase rather than merely mirror parity.

9.4 Scalarisation and character loss

Let L be a linear readout invariant under one mirror J_j:

$$LJ_j = L.$$

If v lies in a J_j-odd sector, then $J_j v = -v$ and therefore

$$Lv = LJ_j v = -Lv.$$

Hence

$$\boxed{Lv = 0.}$$

More generally, an invariant scalarisation kills every character which is nontrivial on the subgroup under which that scalarisation is invariant. This gives a representation-theoretic form of the pair-first rule:

$$\boxed{\text{resolve irreducible character sectors before quotienting.}}$$

A simple nonlinear example is the even power quotient

$$(-z)^4 = z^4,$$

which cannot retain a half-turn-odd character. H-VER056, by contrast, proves for its relative-return observable ν that

$$\nu(-q) = -\nu(q), \qquad \nu(q^{-1}) = \nu(q).$$

Therefore, with character order (h, X), its transformation law is

$$\boxed{(-,+),}$$

with canonical projector

$$\Pi_{-,+} = \frac{1}{4}(I - h)(I + X).$$

This statement types the observable's transformation law; it does not assert that ν is itself a group homomorphism.

9.5 Commuting conjugate norms

For commuting involutions J_1, J_2 on a commutative algebra, define

$$N_J(f) = fJ(f).$$

Then

$$\begin{aligned} N_{J_2} N_{J_1}(f) &= f(J_1 f)(J_2 f)(J_1 J_2 f) \\ &= N_{J_1} N_{J_2}(f). \end{aligned}$$

The four-corner orbit

$$(f, J_1 f, J_2 f, J_1 J_2 f)$$

nevertheless contains strictly more labelled character information than the single iterated norm product. Conjugate norm closure is therefore a downstream scalarisation, not a replacement for the pair or character cube.

This generic statement accommodates complex pairs such as $A \pm iB$ and golden Galois pairs such as $P_\Phi, P_{\sigma\Phi}$. A third Galois mirror may be promoted to a C_2^3 example only after its action on the same source object and the commutators with h and X have been proved. No H_4, E_8, Coxeter, O10, or Riemann identification is inferred by this generic character-cube theorem.

10 Dihedral lift and the hypergroup boundary

Let

$$D_N = C_N \rtimes C_2 = \langle r, s \mid r^N = s^2 = 1,\ srs = r^{-1} \rangle.$$

The cyclic power map lifts to

$$\Delta_m(r) = r^m, \qquad \Delta_m(s) = s.$$

The defining relation is preserved, so Δ_m is an endomorphism and

$$\boxed{\operatorname{Im} \Delta_m = \langle r^{\gcd(N,m)}, s \rangle.}$$

Its image order is

$$\frac{2N}{\gcd(N,m)},$$

and

$$\Delta_a \circ \Delta_b = \Delta_{ab}.$$

Thus the cyclic kernel/image grammar lifts to a recursive dihedral subgroup grammar before radialization.

This structure mirrors the repeated subgroup factorisations used in the repository's affine/dihedral D_{12} constructions. The word "mirrors" is important: a generic D_{24} image subgroup is not, by that fact alone, the separately typed repository D_{12} object. Rotation convention, quotient, and source role remain load-bearing.

The hypergroup layer is downstream. Classical Chebyshev hypergroups arise from the nonnegative product linearization of Chebyshev polynomials, while the HG repository also realizes first-kind Chebyshev structure as a radial/double-coset algebra. Group subgroups, coset torsors, and exact-order root strata must therefore not be silently replaced by one another after the hypergroup quotient.

This is also the right place to read the H-VER057 information-loss obstruction. If a quotient kills a central half-turn but the required target observable detects it, the target map cannot factor through the quotient. Pair-first transport must occur before the orientation bit is lost.

11 Literature and prior-art boundary

The ingredients of the present construction have substantial classical literature. The Stacks Project records the standard principal homogeneous-space definition: a torsor is characterized by a simply transitive group action. Standard abstract algebra supplies the first isomorphism theorem and the correspondence between the subgroup lattice of a cyclic group and the divisor lattice of its order. In particular, standard C_{24} subgroup diagrams contain the order-three subgroup alongside the orders $2, 4, 6, 8, 12$ and 24.

The same classical cyclic-group structure also controls the common-clock construction of Section 6. For finite cyclic subgroups of orders a and b, the intersection has order $\gcd(a, b)$ and the subgroup they generate has order $\operatorname{lcm}(a, b)$. The Stacks Project also records the roots-of-unity functor μ_n as a group object, providing a standard algebraic setting for these finite root-of-unity groups. Thus the C_{60} and C_{120} parent calculations are applications of classical cyclic-group and roots-of-unity algebra rather than new group theory.

Cyclotomic polynomials are classically defined by primitive exact-order roots and satisfy

$$x^n - 1 = \prod_{d|n} \Phi_d(x).$$

Standard references also record the prime substitution rules

$$\Phi_{np}(x) = \Phi_n(x^p) \quad (p \mid n),$$

and

$$\Phi_{np}(x) = \frac{\Phi_n(x^p)}{\Phi_n(x)} \quad (p \nmid n).$$

The exact-order pullback formula of Section 4 packages these familiar identities directly in terms of the order transformation $r \mapsto r/\gcd(r, m)$. In particular, the standard prime substitution rule with $p = 2$ and $n = 5$ gives

$$\Phi_5(x^2) = \Phi_5(x)\Phi_{10}(x),$$

matching the executable pentic pullback retained here.

The hypergroup literature is likewise well established. Jewett's abstract convolution framework is foundational; Bloom and Heyer give a systematic harmonic-analysis treatment. Lasser develops polynomial hypergroups from orthogonal polynomials. Tsurii explicitly treats the Chebyshev hypergroups of first and second kind and their q-deformations, including the reciprocal expression

$$T_n\left(\frac{q + q^{-1}}{2}\right) = \frac{q^n + q^{-n}}{2}.$$

Wildberger develops finite commutative hypergroups with applications to group theory and conformal field theory, while Hermann studies hypergroup homomorphisms in representation-theoretic settings.

The literature search performed for this paper supports all of those component facts. It did not locate, in the sources examined, a single reference which packages the following complete chain as one theorem:

basepoint-independent torsor transport $\to$ induced homomorphism
$\to$ recursive kernel/image and torsor fibres
$\to$ exact-order pullback strata $\to$ cyclotomic products,

with the H-VER037 pair-first Rule/Rock packet and the half-turn information-loss criterion retained simultaneously. This is a literature-search boundary, not a claim that no such formulation exists. The generic algebraic components are classical; the executable synthesis and its source-typing discipline are the contribution recorded by the HG theorem chain.

12 Application guidance and open boundary

The generic theorem suggests a strict workflow for downstream use. First retain the oriented source routes. Second prove a natural torsor map between the relevant route systems. H-VER057 then supplies the induced group homomorphism. Third prove injectivity on the relevant relative-return subgroup; equalizer reflection follows. Only after those steps may one descend to scalar returns or quotient coordinates.

A scalar quotient which has already killed a half-turn cannot be used to construct a target return that is odd under that half-turn. The correct object to intertwine is the oriented route diagram before quotienting, not a scalar return from which orientation has disappeared.

Likewise, the C_{24} cyclotomic products do not by themselves identify an application route. They provide a canonical generic lattice of subgroups, kernel torsors, and exact-order strata. A downstream theory must prove that its source objects occupy the required nodes and arrows of that lattice.

The common C_{60}/C_{120} parent likewise supplies an ambient finite closure only. The facts that C_{24} and C_{40} are coordinate faces of C_{120} and share C_8 do not identify a downstream triadic route with a pentic route. Such a theory must still supply the source-owned map between the corresponding oriented diagrams.

The following statements remain outside this paper:

$$\begin{aligned} \text{one generic } D_{24} \text{ image subgroup} &= \text{ repository affine } D_{12}, \\ \text{one cyclotomic exact-order stratum} &= \text{ one application-specific carrier}, \\ \text{one scalar return homomorphism} &= \text{ one source-role intertwiner}. \end{aligned}$$

No proof of the Riemann hypothesis is claimed. The value of the present paper for such a downstream programme is narrower: it removes the need to re-prove generic torsor, subgroup, fibre, and cyclotomic algebra, and makes explicit the information that must still be transported before a quotient is taken.

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