The Two-Stage Arithmetic Projector on the Bilateral De-Vries Jordan Nilpotent Tower

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Abstract

On the multiplicative dilation monoid the logarithm of the full number line, expanded over ordered factorizations with harmonic weights and no primes, cancels every integer with two or more distinct prime factors and leaves weight $1/a$ on each prime power p^a. Perfect-power Möbius inversion then leaves the primes, and the convolution exponential returns the full line. Represented on the bilateral De Vries–Jordan–Jacobi (DVJ) alpha tower, a two-sided Gaussian character lattice, the hierarchy becomes an Euler-log theta and a prime theta, absolutely convergent and entire in the tower index; on two independent legs it becomes a Dirichlet-convolution conductor packet. The odd sublattice of the half-index tower, the triangular DVJ family, is, up to an elementary factor, the tower itself half a quasi-period away: two triangular sheets on two reciprocal rails, the two halves of a global half-character theta; their literal sheet presentations share one endpoint, subtracted once. The frozen DVJ data enter only as downstream specializations, with the negative tail of the lattice kept. Classical analytic readouts are recorded behind a firewall. The arithmetic identities are classical and are credited as such; the contribution is the source order, full line first, and the bilateral realization. The results are owned by theorems of the Hypergroup project and are scalar: lifting them to all forty labels of the project's full forty-label source is stated as an open aim, not proved. No Riemann-hypothesis claim is made.

Contents

From the Full Number Line to the Primes 4

1 Scope, separation and non-claims 5
- 1.1 The question and the three-paper series . . . 5
- 1.2 Full-line premise and classical arithmetic boundary . . . 5
- 1.3 The infinite bilateral representation . . . 6
- 1.4 Ownership and readout firewall . . . 6

2 The dilation monoid and the full number line 7
- 2.1 A formal multiplicative source . . . 7
- 2.2 The full-line element and its augmentation . . . 7
- 2.3 Source versus analytic realization . . . 8

3 The multiplicative logarithm 8
- 3.1 Ordered factorizations before prime support . . . 8
- 3.2 Identifying the surviving support . . . 8
- 3.3 Where the harmonic weight comes from . . . 9
- 3.4 Ownership, comparison and analytic boundary . . . 10

4 Perfect-power Möbius inversion and the inverse 10
- 4.1 Power substitution on the full dilation algebra . . . 10
- 4.2 The divisor sum and cancellation of higher powers . . . 11
- 4.3 Convolution exponential recovers the full line . . . 11
- 4.4 Ownership and scope of the second stage . . . 12

5 Analytic realization: absolute convergence and entireness 12

6 Representation on the bilateral DVJ alpha tower 14
- 6.1 The entire character family before specialization . . . 14
- 6.2 Infinite reciprocal modes and second-order relations . . . 15
- 6.3 Euler-log and prime-theta layers . . . 15
- 6.4 Odd-lattice extraction: the two triangular sheets and their rails . . . 16
- 6.5 The three downstream DVJ specializations . . . 18
- 6.6 The DVJ tower inside the theta lattice . . . 19
- 6.7 The distinct zero-character recognition . . . 20

7 The two-leg conductor representation 21
- 7.1 Two independent bilateral legs . . . 21
- 7.2 One-leg dilations and exact inclusion–exclusion . . . 21
- 7.3 Common heat is not one-leg conductor motion . . . 22
- 7.4 Comparison with the charged two-leg law, and the retained boundary . . . 23

8 Readout comparison, behind the firewall 23
- 8.1 The readout is terminal, not an arithmetic premise . . . 23
- 8.2 The two classical Dirichlet-series comparisons . . . 24
- 8.3 What the scalar readout forgets . . . 24
- 8.4 No reverse ingress or analytic-continuation claim . . . 25

9 In the literature 26

10 Conclusion 29

Appendices 31

A Repository theorem provenance **31**

B Executable receipts and audit boundary **33**

C Equation contract: the two-stage arithmetic projector **35**

D Boundary and open comparisons **43**
D.1 What the paper proves, and who owns it 43
D.2 Comparison register: annotations, never ingress 44
D.3 Negative ledger: what is not proved 44
D.4 What would discharge the open boundary 45

From the Full Number Line to the Primes

This summary of equations is a reader-facing roadmap, not a second dependency DAG (directed acyclic graph); that is Appendix C. It keeps the formal coefficient algebra of the dilation monoid distinct from its analytic realization on the bilateral tower, and it shows where the frozen De Vries–Jordan–Jacobi (DVJ) data enter: at the end, never at the source. Here $\delta_n^\times$ is the dilation by the positive integer n, defined in (1): $\delta_m^\times \delta_n^\times = \delta_{mn}^\times$, $\delta_1^\times = 1$, acting on the tower by $(x, r) \longmapsto (n^2 x, nr)$. Throughout the roadmap, p denotes a prime and $\delta_p^\times$ the corresponding prime dilation. Also, as boxed in Appendix C: $B_r^\alpha(x) = \sum_{k\in\mathbb{Z}} \exp(-\pi x k^2 + rk\ell_t)$, $\ell_t = \log a_t$, $g_r^\alpha(x) = 2\exp(-\pi x)\cosh(r\ell_t)$, $x_0^\triangle = L_\triangle/(2\pi)$, the triangular sheets $T_+(n) = n(n+1)/2$ and $T_-(n) = n(n-1)/2$, the half-character theta $\widehat{\Theta}$ with odd index $h(n) = 2n+1$, and $\sum_k \mu_{\mathrm{M}}(k)\delta_k^\times = \mathcal{N}_\times^{-1}$.

full number line

$$\mathcal{N}_\times = \sum_{n\geq 1} \delta_n^\times = 1 + \mathcal{X}_\times, \qquad \text{every positive integer, weight one}$$

$\Downarrow$ $\log_\star$ over ordered factorizations, weights $(-1)^{k+1}/k$

prime powers

$$\mathcal{L}_\times = \sum_p \sum_{a\geq 1} \frac{1}{a}\, \delta_{p^a}^\times, \qquad \text{mixed prime support cancels exactly}$$

$\Downarrow$ $\sum_{k\geq 1} \frac{\mu_{\mathrm{M}}(k)}{k}\, \psi_k^\times$ (perfect-power Möbius inversion, $\psi_k^\times(\delta_n^\times) = \delta_{n^k}^\times$)

primes

$$\mathcal{P}_\times = \sum_p \delta_p^\times, \qquad \exp_\star(\mathcal{L}_\times) = \mathcal{N}_\times \text{ (and back)}$$

$\Downarrow$ $\delta_n^\times : (x, r) \longmapsto (n^2 x, nr)$ on the bilateral tower

Euler-log theta and prime theta

$$B_r^\alpha(x) - 1 = (\exp_\star(\mathcal{L}_\times)\, g^\alpha)_r(x), \qquad (\mathcal{P}_\times\, g^\alpha)_r(x) = \sum_p 2\, \exp(-\pi p^2 x)\cosh(pr\ell_t)$$

absolutely convergent; entire in r

$\Downarrow$ parity split of the half-index tower (Section 6.6)

triangular family: the odd sublattice, $v = r\ell_t/2$

$$B_{r/2}^\alpha(x_0^\triangle/4) = \underbrace{B_r^\alpha(x_0^\triangle)}_{j \text{ even}} + \underbrace{\widehat{\Theta}(x_0^\triangle, v)}_{j=h(n) \text{ odd}}, \qquad \widehat{\Theta} = \widehat{\Theta}_+ + \widehat{\Theta}_-$$

$$\widehat{\Theta}_+(x_0^\triangle, v) = \sum_{n\geq 0} \exp\Big((2n+1)v - L_\triangle \tfrac{(n+\frac{1}{2})^2}{2}\Big) = \widehat{\Theta}_-(x_0^\triangle, -v),$$

$$\widehat{\Theta}_-(x_0^\triangle, v) = \sum_{n\geq 0} \exp\Big(-(2n-1)v - L_\triangle \tfrac{(n-\frac{1}{2})^2}{2}\Big) - \exp\big(v - L_\triangle/8\big) = \widehat{\Theta}_+(x_0^\triangle, -v)$$

two sheets: $\frac{(n+\frac{1}{2})^2}{2} = T_+(n) + \frac{1}{8}$, $\frac{(n-\frac{1}{2})^2}{2} = T_-(n) + \frac{1}{8}$, $T_-(n) = T_+(n-1)$

$\Downarrow$ downstream only: $L_0^\triangle$, α_{dVJ}, $r \longmapsto 1$

dual rail at index one, $y = \exp v$, $y_t = \exp(\ell_t/2)$

$$K_+^\triangle(y) = y^{-1}\exp(L_\triangle/8)\,\widehat{\Theta}_+(x_0^\triangle, v), \quad K_-^\triangle(y) = y^{-1}\exp(L_\triangle/8)\,\widehat{\Theta}_-(x_0^\triangle, v)$$

$J_1^\triangle = K_+^\triangle(y_t) + K_-^\triangle(y_t)$: frozen DVJ defect, with the negative rail kept

Two branches leave this chain and feed nothing back into it. The readout $\delta_n^\times \longmapsto n^{-s}$ gives the classical Euler-product and prime-zeta formulas on a half-plane; it is lossy and behind a firewall (Section 8). The lift of the whole chain to all forty labels of the project's full forty-label (FULL40) source is the open aim of the paper, stated and not proved (Appendix D).

1 Scope, separation and non-claims

1.1 The question and the three-paper series

This is Paper C (H-PAP018) of a three-part investigation into the bilateral De Vries–Jordan–Jacobi (DVJ) source and its arithmetic and entire-function readouts. Paper A (H-PAP016 [55]) investigates the source-preserving DVJ-to-Ξ-ENTIRE connection. Paper B (H-PAP017 [56]) studies the dyadic and weighted-dilation Euler bridge. The present paper asks a different, prior arithmetic question: *can the full positive-integer source produce prime-power and prime support by exact cancellation, without beginning with a prime-supported selection?*

The three papers are companion investigations, not three steps in an assumed proof of the Riemann hypothesis. In particular, no conclusion of Paper A or B supplies a load-bearing premise here. All mathematical assertions made in this paper must be justified by the present source algebra, its declared theorem owners, and the analytic representation stated below. Papers A and B are compared with this one but never used as premises: no result of either is a dependency of any result here, and where this paper mentions one of them, the mention is a cross-reference for the reader, not a step of an argument.

1.2 Full-line premise and classical arithmetic boundary

The starting object is the complete multiplicative dilation monoid of box (37), together with the full-line element $\mathcal{N}_\times$ and its augmentation $\mathcal{X}_\times$ in box (38). Its elements are the formal dilations $\delta_n^\times$, one for each positive integer n, with $\delta_m^\times \delta_n^\times = \delta_{mn}^\times$ and $\delta_1^\times = 1$ (Section 2). Throughout, the letter p denotes a prime and $\delta_p^\times$ the dilation at a prime; primes enter only as the support that the stages below produce, never as an input. Every positive integer is present with coefficient one. Neither a prime subset nor the answer of a primality test appears in this initial definition.

The first stage expands the formal convolution logarithm over ordered factorizations, as in box (39). Harmonic weights cancel all mixed-prime support, leaving coefficient $1/a$ on each prime power; this is Linnik's identity [10, 11, 12], and the result and its classical Euler-product comparison [1, 4, 5] are stated in box (40). Perfect-power Möbius inversion then cancels the higher powers, as in Riemann's inversion of his prime-power counting function [1, 2], leaving the prime-supported element in box (41). The convolution exponential of box (42) recovers the full-line element, the integer case of Beurling's relation between generalized primes and integers [20, 21]. Section 9 discusses this prior art.

Here "no primes as an initial premise" refers to the *construction and support of the initial source*, not to a purported derivation of unique factorization from nothing. The proof identifying the surviving coefficients uses classical unique factorization, the Euler product and Möbius inversion [4, 3, 6]; this paper claims no priority over those arithmetic identities. Likewise, "projector" names a two-stage arithmetic support-extraction construction. It does not assert that its logarithm and inversion form a single linear idempotent projector on an unspecified normed operator space. Coefficientwise formal identities must not be promoted into analytic operator identities without a declared domain and a separate argument.

1.3 The infinite bilateral representation

The paper's additional mathematical structure is the action of these coefficients on an *entire family* of bilateral Gaussian-character carriers, rather than on one frozen scalar. The source seed g^{α}, character base a_{t}, and bilateral tower B^{α} are registered in box (43). Their dilation acts simultaneously on Gaussian scale and character index. The analytic justification is the locally uniform, absolute convergence proved in Lemma 5.1, under the source contract box (44); its domain is positive Gaussian scale and complex character index. The Euler-log theta and prime theta are the two different coefficient actions of boxes (45) and (46).

The nested infinity here matters: the full multiplicative series acts on a bilateral lattice whose reciprocal character cells already form an infinite family. Each paired cell has its own Chebyshev second-order recurrence; no single fixed-coefficient recurrence for the summed tower is inferred from that fact. The triangular odd-lattice family $J^{\triangle}$ in box (47) makes this source-retaining structure explicit: it is made of the two triangular sheets $T_{+}(n) = n(n+1)/2$ and $T_{-}(n) = n(n-1)/2 = T_{+}(n-1)$, each on its own reciprocal rail, $K_{+}^{\triangle}$ and $K_{-}^{\triangle}$ (Section 6.4). Its frozen scale, the canonical DVJ root data, and the special character member at index one occur *after* the family has been constructed: boxes (51), (52), (53), and (54). The frozen equation retains the negative-rail correction, the bracket $y_{\mathrm{t}}^{2}K_{-}^{\triangle}(y_{\mathrm{t}}) - 1$ built from the lower-sheet rail $K_{-}^{\triangle}$; the uncorrected bilateral scalar is not substituted for the DVJ closure equation. The zero-character comparison of box (55) is a separate recognition branch, never an input selecting the DVJ root.

Two independent lattice legs provide the finite-conductor Dirichlet-convolution packet and the overlap-subtraction law of box (50). The shared prime-divisor overlap has a greatest-common-divisor (GCD) interpretation; it does not license importing charged or route-valued carriage that the scalar packet has not established. The detailed two-leg argument belongs to Section 7.

1.4 Ownership and readout firewall

The full-integer cancellation and inverse are owned by H-VER174; the bilateral analytic action by H-VER175; the triangular specialization and corrected DVJ insertion by H-VER176; and the two-leg scalar law by H-VER173-A, with the source dilation monoid in H-VER040. These are closed *source-scalar (SOURCE-SCALAR)* results. Their proofs are not invalidated by their placement outside the separately certified retained full forty-label source (FULL40): the forty labelled Jacobi source imported from H-PAP015 [54]. A FULL40 lift would transport the scalar coefficients coherently to all forty retained labels; that transport is open, as recorded in box (57) and Appendix D. Conversely, closing scalar results does not prove such a lift. The open transport is associated with H-VER172 and is not assumed in this paper.

The Dirichlet-series readout box (56) is terminal, domain-restricted and information-losing (owner ruling M1, the paper's loss-of-information firewall). Its classical $\log\zeta$ comparison [5] is not an inverse reconstruction of the bilateral character rails, a proof of analytic continuation, or a source premise. In particular, the paper does not establish a spectral zero correspondence for the frozen DVJ root, force any zeros of Ξ onto the critical line, prove the Riemann hypothesis, or claim that the FULL40 charged/route occupancy follows automatically. Those boundaries are carried forward explicitly to Appendix D.

The subsequent sections prove the formal coefficient identities first, justify the analytic series separately, and only then examine the DVJ and two-leg realizations. No negative result about an

unconstructed FULL40 lift is implied by this source-scope limitation.

2 The dilation monoid and the full number line

2.1 A formal multiplicative source

The starting index set is the *entire* multiplicative monoid of positive integers, with identity 1. For each $n \geq 1$ the *dilation* $\delta_n^\times$ is the formal basis element indexed by n: the pointed-lattice source operation $k \mapsto nk$ supplied by H-VER040. Dilations compose as their indices multiply, and on a function $\varphi_r(x)$ of a Gaussian scale x and a character index r, as on the bilateral tower of Section 6, a dilation acts on both together:

$$\delta_m^\times \delta_n^\times = \delta_{mn}^\times, \qquad \delta_1^\times = 1, \qquad (\delta_n^\times \varphi)_r(x) = \varphi_{nr}(n^2 x), \text{ that is } (x, r) \longmapsto (n^2 x, nr). \tag{1}$$

The composition law is box (37); the action on the tower's seed is box (43). No prime-supported submonoid, primality predicate or Euler product is used to define this source. This source-side use of a multiplicative monoid may be compared with the Bost–Connes semigroup action, cyclotomic endomotives, and Euler products over the strong atoms of Krull monoids with torsion class group equipped with a scale [22, 42, 43]; none of those comparisons is used as a premise here.

We work first with formal series having rational coefficients at *every* positive-integer index. The multiplication in box (37) is Dirichlet convolution [3, 6]: the coefficient at a fixed index N is a sum over the finitely many factor pairs with product N. Thus this algebraic multiplication requires no choice of analytic norm, decay condition or convergence of the series as operators. It is commutative because multiplication of positive integers is commutative.

It is also associative. Indeed, for any three coefficient sequences f, g, h, both bracketings of the convolution have the same coefficient at N:

$$\begin{aligned} \sum_{ab=N} \sum_{uv=a} f(u)g(v)h(b) &= \sum_{uvw=N} f(u)g(v)h(w) \\ &= \sum_{ut=N} \sum_{vw=t} f(u)g(v)h(w). \end{aligned} \tag{2}$$

Every sum in (2) is finite, so no rearrangement of an infinite series is involved. The unit is $\delta_1^\times = 1$, as registered in box (37). These are the ring axioms of the arithmetic functions under Dirichlet convolution [3], a ring that Cashwell and Everett showed to be a unique factorization domain [9]. This formal, coefficientwise algebra is the source-side setting of H-VER174, not a claimed bounded operator algebra on a Hilbert space.

2.2 The full-line element and its augmentation

The element $\mathcal{N}_\times$ in box (38) includes *all* positive integers, each with coefficient one. Removing only the multiplicative identity produces $\mathcal{X}_\times$. The series with zero coefficient at $\delta_1^\times$ form an ideal: the coefficient at the identity in a convolution is the product of the two identity coefficients. In particular, $\mathcal{X}_\times$ belongs to that ideal. Nothing in this definition detects or singles out primes.

A useful finiteness bound follows directly from the augmentation's support. Every term in its k-fold convolution power is indexed by a product of k integers each at least two. Therefore

$$[\delta_n^\times]\mathcal{X}_\times^k = 0 \qquad \text{whenever } n < 2^k, \quad k \geq 1. \tag{3}$$

This is the body form of the cutoff already registered in Appendix C. The cutoff (3) means that, for any fixed n, only finitely many powers of the augmentation can contribute to its coefficient; the same cutoff appears in the ordered-factorization form of Linnik's identity [12]. The convolution logarithm displayed in box (39) is consequently a legitimate *formal* series, even though infinitely many powers occur in its notation. Its coefficientwise expansion and the resulting arithmetic cancellation are proved in Section 3; they are not premises of the full-line construction.

2.3 Source versus analytic realization

Two uses of the word "infinite" must remain distinct. The formal algebra above has infinitely many positive-integer dilation indices, but *each coefficient calculation* is finite by box (37) and (3). Later, the same coefficients act on a bilateral Gaussian-character tower with its own infinite lattice sum, as recorded in box (43). Absolute and locally uniform convergence for the specified analytic actions is a separate result of Lemma 5.1; formal coefficientwise finiteness alone does not prove it.

Likewise, extending the multiplicative source operation from a single $\delta_n^{\times}$ to arbitrary formal series is an algebraic bookkeeping construction. No FULL40 transport follows from it. That separately open comparison is stated in box (57). Thus the next section can investigate arithmetic cancellation without importing either analytic convergence or a retained forty-label operator premise.

3 The multiplicative logarithm

3.1 Ordered factorizations before prime support

The source of Section 2 contains every positive integer with coefficient one. We now apply the formal convolution logarithm to this *full* source, without selecting prime indices. The project-registered expression is box (39): its kth term has the harmonic factor $(-1)^{k+1}/k$, multiplying the number $f_k^{\mathrm{ord}}(n)$ of ordered k-factor decompositions of n, with each factor at least two. These are the ordered factorizations studied by Kalmár and Hille [13, 14].

The coefficientwise status of this infinite notation matters. By (3), a fixed coefficient at n receives no contribution once $2^k > n$. Hence the upper limit $k \le \lfloor \log_2 n \rfloor$ in box (39) is finite. No normed operator logarithm, analytic continuation, or interchange of two uncontrolled infinite sums is used at this stage. Associativity of these finite coefficient operations was established in (2).

The coefficient at the identity is zero: every positive power of $\mathcal{X}_{\times}$ has vanishing identity coefficient. We therefore seek the support of this logarithm only among $n \ge 2$. In particular, no occurrence of a prime in the *answer* may be mistaken for a prime-support restriction in the premise.

3.2 Identifying the surviving support

The support identification uses the classical fundamental theorem of arithmetic [4] *after* the full-integer logarithm has been constructed. Unique factorization proves, coefficient by coefficient, the Euler product identity [4, 5] already registered in box (40): for a fixed integer n, only its finitely many prime divisors contribute, and exactly one choice of nonnegative exponents has product n. This argument establishes the product; it does not define the full-line source by presupposing a prime selector. Structurally, using a convolution logarithm to expose primitive content has a close

analogue in the Eulerian idempotent of a connected filtered cocommutative bialgebra over a field of characteristic 0 [36]. The analogy concerns the role of the logarithm only; H-VER174 proves the present arithmetic coefficients in the dilation algebra itself.

Taking the formal convolution logarithm of that identity gives

$$\begin{aligned}\log_\star(\mathcal{N}_\times) &= \log_\star \prod_p (1-\delta_p^\times)^{-1} \\ &= \sum_p \sum_{a\geq 1} \frac{1}{a}\delta_{p^a}^\times = \mathcal{L}_\times.\end{aligned} \tag{4}$$

In (4), the logarithm of a product becomes a sum because the coefficient algebra is commutative over the rationals and all relevant products are finite at each fixed index. The geometric-series factors and the logarithmic expansions are likewise coefficientwise formal. Thus (4) proves the equality of the ordered-factorization logarithm and its classical Euler-log expression; it is not a Dirichlet-series transform readout. Read coefficient by coefficient, this equality is Linnik's identity [10, 11], which Sprittulla states in exactly this ordered-factorization form, with the same cutoff $2^k \leq n$ [12]. No priority is claimed for it; what this paper adds is the reading order, full line first.

Two precise conclusions follow from the coefficient identity in box (40). Every index with two or more *distinct* prime factors has coefficient zero, and every prime power p^a has coefficient $1/a$. Here a is the prime-power exponent, not the integer index of an independent prime selector. The power $a = 1$ survives with coefficient one; the removal of higher powers is the separate second-stage argument in Section 4.

3.3 Where the harmonic weight comes from

The factor $1/a$ can also be seen directly on the original ordered-factorization side. When the *output index* is p^a, an ordered factorization into k nonunit factors is a composition of the exponent a into k positive parts, and the count of these is MacMahon's [15]. Consequently, for $1 \leq k \leq a$,

$$\begin{aligned} f_k^{\mathrm{ord}}(p^a) &= \binom{a-1}{k-1}, \\ \sum_{k=1}^{a} \frac{(-1)^{k+1}}{k} f_k^{\mathrm{ord}}(p^a) &= \frac{1}{a}\sum_{k=1}^{a}(-1)^{k+1}\binom{a}{k} = \frac{1}{a}. \end{aligned} \tag{5}$$

The last equality in (5) is the binomial identity obtained from $(1-1)^a = 0$. Its harmonic denominator is essential: replacing $1/k$ by one does not give the same coefficient. Thus the $1/a$ survives for an exact arithmetic reason, rather than by a numerical fit.

For a mixed-factor illustration, the ordered-factorization counts at 6 are $1, 2$, and those at 360 are $1, 22, 111, 220, 190, 60$. Substitution into the original harmonic series gives

$$\begin{aligned} [\delta_6^\times]\mathcal{L}_\times &= 1 - \frac{2}{2} = 0, \\ [\delta_{360}^\times]\mathcal{L}_\times &= 1 - \frac{22}{2} + \frac{111}{3} - \frac{220}{4} \\ &\quad + \frac{190}{5} - \frac{60}{6} = 0. \end{aligned} \tag{6}$$

The exact finite cancellations in (6) illustrate the general proof, not a separate assumption establishing it. They explain why a full-integer source can reveal prime-power support *without ever initially deleting composite indices.*

3.4 Ownership, comparison and analytic boundary

This source-side result is owned by H-VER174 (checks A2 and A3). Its executable receipt compares the ordered-factorization coefficients with the product-derived coefficients using exact rational arithmetic and checks hostile coefficients. The prime-product Euler-log identity proved by H-VER173-A is a *MATCH*: it reaches this same series from a prime-product representation, but is not a premise of the full-line calculation here. No statement from the still-open H-VER173 is imported as an owner.

The logarithm in box (39) is formal: its coefficientwise finiteness does not alone license an infinite analytic sum after substitution of the bilateral Gaussian-character seed. In particular, one must not use the completed Möbius-stage bound of (14) to dominate the *uncollapsed* ordered-factorization expansion. The separately proved analytic results are stated in Lemma 5.1.

Finally, the coefficient equality (4) does not establish a source-preserving inverse to any zeta or Xi readout, nor a lift to all forty labels of the retained FULL40 source. The prime-power layer is the input to the distinct Möbius power-substitution stage of box (41); nothing from that later elimination stage is used to establish this section.

4 Perfect-power Möbius inversion and the inverse

4.1 Power substitution on the full dilation algebra

Section 3 constructed the Euler-log $\mathcal{L}_\times$ from the complete positive-integer source. Its support consists of prime powers, with coefficient $1/a$ at p^a, as established in box (40) and (4). That is the *output* of the first cancellation, not a support restriction on the original source. The second stage now removes all higher prime powers by a distinct coefficientwise inversion. Witt-vector and necklace-ring constructions provide a useful algebraic analogy in which product data, logarithmic/ghost coordinates and Möbius inversion separate primitive cyclic data from repetitions [37, 38, 39]. The present power substitution and coefficients are nevertheless the source operations proved here, not a Witt-vector import.

For each positive integer k, the power substitution $\psi_k^\times$ sends each dilation to the dilation by the kth power of its index, $\psi_k^\times(\delta_n^\times) = \delta_{n^k}^\times$, and extends linearly to formal series; it is defined in the equation contract, box (41) in Appendix C. It respects the same multiplicative law as the underlying basis, giving the following useful identities:

$$\begin{aligned} \psi_k^\times(\delta_m^\times\delta_n^\times) &= \delta_{(mn)^k}^\times = \psi_k^\times(\delta_m^\times)\psi_k^\times(\delta_n^\times), \\ \psi_j^\times\psi_k^\times(\delta_n^\times) &= \delta_{n^{jk}}^\times = \psi_{jk}^\times(\delta_n^\times). \end{aligned} \tag{7}$$

The relations in (7) hold first on basis elements and extend coefficientwise to the formal completion: for a fixed output index n, there are finitely many pairs of inputs with product n. No normed-space endomorphism or infinite analytic interchange is being asserted by this formal calculation.

4.2 The divisor sum and cancellation of higher powers

Use the Möbius weights $\mu_{\mathrm{M}}(k)$ registered in box (41): the coefficients of the inverse of the full line, $\mathcal{N}_\times^{-1} = \sum_{j\ge 0}(-\mathcal{X}_\times)^j = \sum_{k\ge 1}\mu_{\mathrm{M}}(k)\,\delta_k^\times$, a formal geometric series finite in each coefficient, defined without naming a prime. Since $\mathcal{N}_\times\,\mathcal{N}_\times^{-1} = 1$, their divisor sum is one at the identity and zero at every other positive integer, and by box (42) $\mathcal{N}_\times^{-1} = \exp_\star(-\mathcal{L}_\times)$. Unique factorization recognizes them as the classical Möbius function: zero on integers divisible by a prime square, and on squarefree integers the signed parity of the number of distinct prime factors [3, 6]. The cancellation below uses only the divisor sum.

Define the *second-stage output* by the project-registered expression $\mathcal{P}_\times$, as in box (41). If the index in the first-stage output is p^b, the substitution with exponent k takes it to p^{bk}. Consequently the coefficient at a fixed prime power p^a is

$$\begin{aligned}[\delta_{p^a}^\times]\mathcal{P}_\times &= \sum_{k\mid a}\frac{\mu_{\mathrm{M}}(k)}{k}\frac{1}{a/k}\\ &= \frac{1}{a}\sum_{k\mid a}\mu_{\mathrm{M}}(k) = \begin{cases}1, & a=1,\\ 0, & a>1.\end{cases}\end{aligned} \tag{8}$$

Equation (8) proves the second exact elimination: every higher power cancels, whereas its exponent-one term survives with coefficient one. Every output of a power substitution of a prime power is again a prime power; therefore an index having two or more distinct prime divisors has coefficient zero at both stages. It follows, *without placing a prime-supported selector in the initial source*, that

$$\mathcal{P}_\times = \sum_{k\ge 1}\frac{\mu_{\mathrm{M}}(k)}{k}\psi_k^\times\mathcal{L}_\times = \sum_p \delta_p^\times. \tag{9}$$

This restates the prime-support contract already registered in Appendix C. The last equality in (9) names its *output support*. It is not a definition of the source and does not reverse the order of the two stages. The unique-factorization theorem [4] is used to identify the surviving support, but it is not used to delete composite indices before the logarithm and Möbius operations act.

This second stage is Riemann's: his prime-power counting function gives the weight $1/a$ to p^a, exactly the coefficients of the Euler logarithm, and he recovers the primes from it by Möbius inversion over perfect powers [1, 2]. So the hierarchy full line, prime powers, primes is Riemann's, and (9) is its formal coefficient version; no priority is claimed for it.

For an explicit hostile check, the exponent-six case yields

$$\begin{aligned}[\delta_{p^6}^\times]\mathcal{P}_\times &= \frac{1}{6}\big(\mu_{\mathrm{M}}(1)+\mu_{\mathrm{M}}(2)+\mu_{\mathrm{M}}(3)+\mu_{\mathrm{M}}(6)\big)\\ &= \frac{1}{6}(1-1-1+1) = 0.\end{aligned} \tag{10}$$

By contrast, dropping the signs in (10) produces $4/6$, not zero. The unsigned stage has nonzero coefficients at higher prime powers; its exact coefficient and analytic majorant are displayed separately in (14). That inequality is an analytic control, not the source of (8).

4.3 Convolution exponential recovers the full line

The Möbius elimination is not the inverse of the first logarithm on an arbitrary input: it extracts prime support from the particular Euler-log source. The inverse of that *formal convolution loga-*

rithm, on the other hand, is the registered convolution exponential from box (42). Coefficientwise formal algebra gives

$$\begin{aligned}\exp_\star(\mathcal{L}_\times) &= \prod_p \exp_\star\left(\sum_{a\ge 1}\frac{\delta_{p^a}^\times}{a}\right)\\ &= \prod_p (1-\delta_p^\times)^{-1} = \mathcal{N}_\times.\end{aligned} \tag{11}$$

Here $\exp_\star$ is the convolution-algebra exponential, not an operator exponential on a specified Hilbert space. The first equality in (11) is valid because the logarithmic terms commute and the coefficient of any fixed index sees only finitely many factors and powers. The middle equality is the formal identity $\exp_\star(\sum_{a\ge1} t^a/a) = (1-t)^{-1}$, applied to a single prime dilation $t = \delta_p^\times$. It is the power series of the logarithm [8] at $z = -t$, $\sum_{a\ge1} t^a/a = -\log(1-t)$ for $|t| < 1$, exponentiated; since both sides are convergent power series in t that agree on $|t| < 1$, their coefficients agree, and the identity holds in the ring of formal power series, where $\exp_\star$ is the exponential series.

The resulting coefficient is exactly one at every positive integer, including the identity, recovering box (38). This is the integer case of Beurling's relation between a system of generalized primes and its generalized integers [20, 21]. Its expansion is a sum of non-negative rational coefficients before collection, an important input to the analytic-realization proof in Lemma 5.1. That lemma establishes convergence only after the Gaussian-character substitution under its stated hypotheses; it is not needed for the coefficientwise proof above.

4.4 Ownership and scope of the second stage

The formal support extraction and exponential inversion are owned by H-VER174 (checks A4 and A5, respectively); the prime-side H-VER157 witness is a comparison, not a new source premise. The full-integer origin from box (38), the first cancellation from (4), and the second elimination from (8) are mathematically distinct results with a fixed derivation order.

The title's *projector* denotes this two-stage support-extraction construction. No general linear, idempotent projection on all formal or analytic carriers is claimed. In particular, neither the arithmetic coefficient identities nor their recovery by the formal exponential constructs a source-preserving inverse to any scalar zeta/Ξ readout, or a FULL40 lift under H-VER172. Those are distinct downstream problems. The next section proves the required convergence when the derived arithmetic elements act on the bilateral DVJ Gaussian-character source.

5 Analytic realization: absolute convergence and entireness

Sections 2–4 are identities between formal series in the dilation monoid; every coefficient there is a finite sum, and no convergence is needed. This section is the single point at which the formal algebra is realized as convergent analytic functions on the tower of Section 6 (box (44)). The lemma is the analytic part of H-VER175 (check B5), restated with its proof and coefficientwise scope. The owner is H-VER175; this exposition does not establish a separate paper-owned theorem.

Fix the tower base $a_\mathrm{t} > 0$, so that $\ell_\mathrm{t} = \log a_\mathrm{t}$ is real, as in box (43). The seed already contains both signs of the bilateral lattice, not a chosen one-sided rail. Under the dilation action, recall

$$(\delta_n^\times g^\alpha)_r(x) = 2\,\exp(-\pi n^2 x)\cosh(nr\ell_\mathrm{t}). \tag{12}$$

Lemma 5.1 (Analytic realization)**.** *Let $\varepsilon > 0$ and $M \geq 0$, and let K be the set of (x, r) with $x \geq \varepsilon$ real and $|\operatorname{Re} r| \leq M$ for complex r. Then, uniformly on this (possibly unbounded) set,*

$$\sum_{n\geq 2} |(\delta_n^{\times} g^{\alpha})_r(x)| \leq 2 \sum_{n\geq 2} \exp(-\pi\varepsilon n^2 + |\ell_{\mathrm{t}}| M n) < \infty. \tag{13}$$

Consequently:

1. *(positive stage) the convolution exponential of the Euler-log element, applied to the seed, has an absolutely and uniformly convergent collected dilation expansion on K and equals the full-line element applied to the seed;*
2. *(completed Möbius stage) the two-stage prime-support realization on the seed converges absolutely and uniformly on K; it equals the sum of the prime dilates;*
3. *for each $x > 0$ both are entire functions of r.*

Proof. The majorant. For complex $w = u + iv$, $|\cosh w|^2 = \sinh^2 u + \cos^2 v \leq \cosh^2 u$. Since ℓ_{t} is real, the real part of $nr\ell_{\mathrm{t}}$ is $n\ell_{\mathrm{t}} \operatorname{Re} r$, so $|\cosh(nr\ell_{\mathrm{t}})| \leq \cosh(n\ell_{\mathrm{t}} \operatorname{Re} r) \leq \exp(n|\ell_{\mathrm{t}}|M)$; with $x \geq \varepsilon$ this gives (13) term by term. The seed terms are as in (12). The majorant does not depend on $(x, r) \in K$, and it converges because the quadratic term dominates the linear one: for $n \geq n_0 = \lceil 2|\ell_{\mathrm{t}}|M/(\pi\varepsilon) \rceil$ one has $|\ell_{\mathrm{t}}|Mn \leq \pi\varepsilon n^2/2$, so the n-th term is at most $\exp(-\pi\varepsilon n^2/2) \leq \exp(-\pi\varepsilon n/2)$, a term of a convergent geometric series, and the finitely many terms with $n < n_0$ are finite. Every term of the dilation series is bounded, at every point of K, by the corresponding term of this one convergent series of constants; so the tail beyond any N is bounded on all of K by the tail of the majorant, which tends to zero. This is Weierstrass's M-test [30], and it gives the uniform convergence on K.

Why this holds for every index. No step uses a truncation or a computed value. Each inequality is proved for an arbitrary index $n \geq 2$, and below for an arbitrary exponent $a \geq 1$; the only infinite object that has to be controlled is the single convergent series of constants on the right of (13). The receipts listed after the proof test the hypotheses and the scope at finitely many indices; the proof does not depend on them.

For item 1: the Euler-log element has the non-negative coefficients $1/a$ at the prime powers p^a, so at each n the convolution exponential has a finite expansion into non-negative rational contributions, and their sum is 1 (box (42); this is the integer case of Beurling's relation [20, 21]). Applied to the seed, this is a double series over the contributions and the index n. For a double series of non-negative terms the order of summation does not change the sum, finite or infinite: this is Tonelli's theorem for counting measure [31]. Taking absolute values of the dilated seed and summing the contributions first gives $|(\delta_n^{\times} g^{\alpha})_r(x)|$ at each n; summing over n then gives the $n = 1$ seed plus the left side of (13), which is finite. So the double series converges absolutely, and Fubini's theorem [31] lets its complex terms be summed in either order with the same value: collecting by n gives the full-line element applied to the seed. This justifies collecting or uncollecting the *positive exponential* only. It does not license rearrangement of the preceding alternating ordered-factorization logarithm, whose absolute coefficients are not bounded by 1 (see the Scope paragraph below).

For item 2: before the signs cancel, the Möbius stage puts on $\delta_{p^a}^{\times}$ the absolute weight in (14):

$$\frac{1}{a} \sum_{k|a} |\mu_{\mathrm{M}}(k)| = \frac{2^{\omega_{\mathrm{HP018}}(a)}}{a} \leq 1. \tag{14}$$

Here $\omega_{\mathrm{HP018}}(a)$ counts the distinct prime divisors of the *exponent* a, not the factors of p^a. The terms $|\mu_{\mathrm{M}}(k)|$ count the squarefree divisors of a, and these number $2^{\omega_{\mathrm{HP018}}(a)}$ [7]. Each distinct prime divisor of a is at least 2, so their product is at least $2^{\omega_{\mathrm{HP018}}(a)}$, and it divides a. So $2^{\omega_{\mathrm{HP018}}(a)} \le a$ for every $a \ge 1$, including $a = 1$, where both sides are 1: one inequality for all exponents at once, not a check of finitely many. This proves the coefficient-one bound in (14). It puts no weight on any n that is not a prime power. By unique factorization [4], distinct pairs (p, a) give distinct $n = p^a \ge 2$, so the absolute series is again bounded by (13). Tonelli's and Fubini's theorems [31] then allow the signed terms to be collected at each p^a, where the Möbius divisor sum $\frac{1}{a}\sum_{k|a} \mu_{\mathrm{M}}(k)$ is 1 for $a = 1$ and 0 for $a > 1$ [6]; what remains is the sum of the prime dilates. In the hostile exponent-six case, the unsigned weight is $4/6$, while the signed coefficient vanishes by (10).

For item 3: each dilated seed is an exponential in x times a hyperbolic cosine in r, so it is entire in r. Fix $x > 0$ and take $\varepsilon = x$. Every compact subset of the complex r-plane lies in a strip $|\operatorname{Re} r| \le M$, so (13) gives uniform convergence on compact sets. By Weierstrass's theorem a locally uniform limit of holomorphic functions is holomorphic, and the derivatives of the partial sums converge locally uniformly to the derivatives of the limit [31]. So both collected series are entire in r, and their character derivatives may be taken term by term: differentiating a term a fixed number of times adds at most a fixed power of n (and a factor depending on $|\ell_{\mathrm{t}}|$), still dominated by Gaussian decay. This justifies the derivatives of these *two realized series*, not differentiation under the whole-line Fourier integral of H-PAP005 [57]. □

Scope. The lemma is stated for the completed Möbius stage and the positive exponential stage. It is not cited for the uncollapsed expansion of the logarithm over ordered factorizations: there the absolute coefficients exceed 1 (at $n = 2^{10}$ their sum exceeds 100), and that stage keeps its own coefficientwise argument (Section 3). The hypothesis that ℓ_{t} is real is used: for complex ℓ_{t} the bound on $\operatorname{Re} r$ no longer controls the hyperbolic cosine.

Receipts. In `docs/audits/dvj-xi-bridge-witness/` and `tests/`; the hostile controls exercise each hypothesis and the scope:

- `witness6_analytic_realization_majorant.py`;
- `test_dvj_xi_bridge_witness6.py`;
- `test_h_pap018_signed_majorant.py` (exact coefficients);
- `src/d12rg_hypergroup/h_ver175_theta_tower_realization.py`, the owner's receipt (check B5, the lemma).

6 Representation on the bilateral DVJ alpha tower

6.1 The entire character family before specialization

The formal arithmetic projection of Sections 2–4 becomes an analytic action only after specifying a representation. The starting point is the *generic* source in box (43): the positive base a_{t}, its real logarithm ℓ_{t}, a positive Gaussian scale x, and a complex character r. The source is not constructed by setting the base equal to the De Vries value. In particular, no choice of a DVJ root is needed to define g^{α} or B^{α}.

The bilateral sum in box (43) contains the zero lattice cell once and every positive and negative nonzero index: it is Jacobi's third theta function at imaginary argument with a character [24]. Pairing indices n and $-n$ gives the hyperbolic-cosine seed without discarding either orientation. The joint dilation $\delta_n^{\times}$ changes *both* the Gaussian argument and the character index. The relation $\delta_m^{\times}\delta_n^{\times} = \delta_{mn}^{\times}$ of box (37) therefore holds in this analytic representation, not only in the formal coefficient algebra. The $n = 1$ seed is already a paired object; there is no independent prime, one-sided, or positive-index source assumption.

6.2 Infinite reciprocal modes and second-order relations

For a fixed positive lattice index n, the two reciprocal characters are obtained from the signed exponents in box (43). The elementary hyperbolic-cosine addition law gives the exact two-step recurrence

$$\begin{aligned} g^{\alpha}_{n(r+1)}(n^2x) + g^{\alpha}_{n(r-1)}(n^2x) \\ = 2\cosh(n\ell_{\mathrm{t}})\, g^{\alpha}_{nr}(n^2x), \qquad n \geq 1. \end{aligned} \tag{15}$$

As a non-load-bearing remark, the two reciprocal branches, not a chosen scalar orientation, produce (15). Its coefficient depends on n. Thus the complete sum of modes is *not* asserted to satisfy one scalar fixed-coefficient recurrence in r. This point matters after the infinitely many arithmetic dilations are substituted into the already infinite bilateral Gaussian lattice.

There is nevertheless one common differential relation for the entire family. This identity is a reader-facing consequence of the already registered tower definition, not a new load-bearing theorem. For character differentiation, each derivative contributes at most a fixed polynomial factor in the lattice index on compact subsets of the character plane. For scale differentiation, one obtains one further quadratic lattice factor. In both cases the resulting polynomial factor is dominated directly by the Gaussian decay for every compact subset of $x > 0$. Uniform convergence therefore permits termwise differentiation. Hence

$$\partial_r^2 B_r^{\alpha}(x) = -\frac{\ell_{\mathrm{t}}^2}{\pi}\partial_x B_r^{\alpha}(x), \qquad x > 0. \tag{16}$$

As a non-load-bearing remark, Equation (16) is the second-order relation shared by the lattice modes; it is the heat equation of the theta functions [26], written in the tower's variables. It is a termwise consequence of the registered tower definition, not a new upstream theorem premise. The paired source remains explicitly available before any scalar evaluation; the differentiated identity alone does not reconstruct its two rails from a scalar value. Both the recurrence (15) and the heat relation (16) hold for the generic base, without invoking the frozen DVJ equation.

6.3 Euler-log and prime-theta layers

The first arithmetic stage, proved under H-VER174, has the formal coefficients $1/a$ at prime powers. Its action on the generic source is the Euler-log theta of box (45). Each summand is the *joint* dilation by p^a, hence has Gaussian exponent quadratic in p^a and character exponent linear in p^a. The second arithmetic stage applies the signed perfect-power inversion of box (41) and gives the prime theta of box (46).

These are two distinct ways of collecting the same registered reciprocal lattice modes: the Euler-log layer carries prime-power harmonic weights, while the completed second stage has prime support

only. Neither layer assumes its prime-supported output when defining the full positive-integer input box (38). The arithmetic cancellation was proved coefficientwise in Sections 3 and 4; it was not inferred from the analytic value of a theta series.

There is an infinite arithmetic series and an infinite signed Gaussian-character lattice, but this is *one* bilateral source with coefficient actions, not a sum of independent complete copies of B^α for each prime. The unpaired zero cell remains 1 in the full tower and is absent from the nonzero seed expansions. H-VER175 and the majorant (13) justify the completed Euler-log and prime-supported series, locally uniformly in complex r. They do not justify an arbitrary rearrangement of the alternating ordered-factorization logarithm before coefficient collection. This separation makes the infinite-in-infinite substitution well-defined rather than merely formal. The dyadic contribution $p = 2$ is a cross-reference to H-PAP017 [56] (Paper B), not an upstream premise or a replacement for the all-integer construction here.

6.4 Odd-lattice extraction: the two triangular sheets and their rails

H-VER176 supplies the exact odd-lattice extraction recorded in box (47). The family is Ramanujan's two-variable theta function $f(A, B)$ evaluated on the tower's character [28], and the extraction is the even and odd split that produces the second Jacobi theta function [24]. Split the integers j of the half-index tower $B^\alpha_{r/2}(x_0^\triangle/4)$ by parity. Since $\pi x_0^\triangle = L_\triangle/2$, its j-th term is $a_{\mathrm{t}}^{rj/2}\exp(-L_\triangle j^2/8)$. The even indices $j = 2m$ give the tower $B^\alpha_r(x_0^\triangle)$ again, and the odd indices give the triangular family:

$$B^\alpha_{r/2}(x_0^\triangle/4) = \underbrace{B^\alpha_r(x_0^\triangle)}_{j\ \text{even}} + \underbrace{a_{\mathrm{t}}^{r/2}\exp(-L_\triangle/8)\,J^\triangle_r}_{j\ \text{odd}}. \tag{17}$$

This is the bracket form of the box, rearranged.

The odd indices fall on two triangular sheets. H-PAP004 [53] owns them (H-VER032; its box is imported in Appendix C):

$$\begin{aligned} T_+(n) = \frac{n(n+1)}{2}, \qquad T_-(n) = \frac{n(n-1)}{2} = T_+(n-1), \\ T_+(n) + T_-(n) = n^2, \qquad T_+(n) - T_-(n) = n. \end{aligned} \tag{18}$$

The two sheets are the inverse Hadamard transform of the tower's own pair of exponents: n^2, the Gaussian exponent, and n, the character exponent. The lower sheet is the upper sheet one step behind: the balance $T_-(n) = T_+(n-1)$ is what pairs the two rails below. Completing the square on each sheet gives the two opposite Gaussian half-shifts,

$$\frac{(2n+1)^2}{8} = T_+(n) + \frac{1}{8} = \frac{(n+\frac12)^2}{2}, \qquad \frac{(2n-1)^2}{8} = T_-(n) + \frac{1}{8} = \frac{(n-\frac12)^2}{2}, \tag{19}$$

the two Gaussian half-rails of H-PAP002 [52]. So the positive odd indices $j = 2n+1 \geq 1$ carry the upper sheet and the negative odd indices $j = -(2n-1) \leq -1$ carry the lower sheet. Taking out the common factor $a_{\mathrm{t}}^{r/2}\exp(-L_\triangle/8)$ leaves the two sums of box (47), each written from $n = 0$, so that the lower one also starts at $j = 1$:

$$J^\triangle_r = \underbrace{\sum_{n\geq 0} a_{\mathrm{t}}^{rn}\exp\bigl(-L_\triangle T_+(n)\bigr)}_{\text{upper sheet, } j=2n+1\geq 1} + \underbrace{\sum_{n\geq 0} a_{\mathrm{t}}^{-rn}\exp\bigl(-L_\triangle T_-(n)\bigr)}_{\text{lower sheet, } j=-(2n-1)\leq -1} \underbrace{-1}_{\text{shared endpoint}}. \tag{20}$$

This accounts for the factors $a_{\mathrm{t}}^{-r/2}$ and $\exp(L_\triangle/8)$ in the bracket form of the box. Both sheets run over $n \geq 0$. Since $T_+(0) = T_-(0) = 0$, both begin at the same point, $j = 1$, whose term is 1: the two sheets share that endpoint, and the lower sheet's own points $j \leq -1$ start at $n = 1$. The -1 is the overlap correction: it subtracts the shared endpoint once, the endpoint of the one-step offset between the sheets, standing as its own term. The sheets are literally the two exponents of Ramanujan's function, $f(A,B) = \sum_{n\in\mathbb{Z}} A^{T_+(n)}B^{T_-(n)}$ [28], and $J_r^\triangle = f\big(a_{\mathrm{t}}^r \exp(-L_\triangle), a_{\mathrm{t}}^{-r}\big)$. Ramanujan's Entry 19, Jacobi's triple product [28], writes the same function as a product, $f(A,B) = \prod_{n\geq 0}(1+A(AB)^n)(1+B(AB)^n)(1-(AB)^{n+1})$. Here $AB = q := \exp(-L_\triangle)$, so

$$\begin{aligned} J_r^\triangle &= \prod_{n\geq 0}(1+a_{\mathrm{t}}^r q^{n+1})(1+a_{\mathrm{t}}^{-r}q^n)(1-q^{n+1}) \\ &= (1+a_{\mathrm{t}}^{-r})\prod_{n\geq 0}(1+a_{\mathrm{t}}^r q^{n+1})(1+a_{\mathrm{t}}^{-r}q^{n+1})(1-q^{n+1}). \end{aligned} \tag{21}$$

Both products run over $n \geq 0$, and their exponents are the steps of the two sheets, $T_+(n+1) - T_+(n) = n+1$ and $T_-(n+1) - T_-(n) = n$: the one-step offset again. The lower product's $n = 0$ factor, $1 + a_{\mathrm{t}}^{-r}$, stands apart as its own factor; what remains is symmetric under $r \mapsto -r$, and since $1 + a_{\mathrm{t}}^r = a_{\mathrm{t}}^r(1 + a_{\mathrm{t}}^{-r})$ it gives, by a second route, the reflection identity (22) proved below.

As a sum or as a product, $J_r^\triangle$ is a *bilateral* function of complex r for every generic positive scale. Replacing r by $-r$ turns each sheet into the other, one step along, because $T_-(n+1) = T_+(n)$; this proves the source-retaining reflection identity

$$J_{-r}^\triangle = a_{\mathrm{t}}^r J_r^\triangle. \tag{22}$$

Both reciprocal rails lie in this odd sublattice, one on each sheet. In the character variable y they are

$$\begin{aligned} K_+^\triangle(y) &= \sum_{n\geq 0} y^{2n}\exp(-L_\triangle T_+(n)) \qquad\quad = y^{-2}K_-^\triangle(y^{-1}), \\ K_-^\triangle(y) &= \sum_{n\geq 0} y^{-2n}\exp(-L_\triangle T_-(n)) - 1 = y^{-2}K_+^\triangle(y^{-1}), \\ J_r^\triangle &= K_+^\triangle(a_{\mathrm{t}}^{r/2}) + K_-^\triangle(a_{\mathrm{t}}^{r/2}). \end{aligned} \tag{23}$$

Both rails run over $n \geq 0$, each on its own sheet. The lower sheet's $n = 0$ term is the shared endpoint 1, since $T_-(0) = 0$, and it is subtracted once, as its own term, as in $J_r^\triangle$ (20). The reciprocity is the balance again: after the endpoint, put $n = m+1$ and use $T_-(m+1) = T_+(m)$, which makes the lower rail $y^{-2}K_+^\triangle(y^{-1})$ term by term; $y \mapsto y^{-1}$ gives the upper line. The factor y^{-2} is the receipt of the one-step displacement between the sheets, not a normalization. The index r enters the family only through the rail variable $y = a_{\mathrm{t}}^{r/2}$, and the reflection $r \mapsto -r$ is $y \mapsto y^{-1}$, which by the reciprocity exchanges the two rails up to the factor y^2; this is (22) in rail form, and as a non-load-bearing remark it is the reflection that Ramanujan's Entry 18 gives for $f(A,B)$ [28]. It is not an instruction to replace those rails by a modulus, a single scalar zero, or an independent selection of the DVJ root.

In the half-index tower the upper sheet sits at $j = 2n+1$ and the lower sheet at $j = -(2n-1)$, so the exchange of the rails is $j \mapsto -j$: the odd-sector form of the same lattice reflection that pairs the two orientations $\pm m$ of the even tower $B_r^\alpha(x_0^\triangle)$. The weight a_{t}^r in (22) is that reflection seen through the factor $a_{\mathrm{t}}^{-r/2}$ of the triangular box: the odd part itself, $a_{\mathrm{t}}^{r/2}J_r^\triangle$, is invariant under $r \mapsto -r$.

The two rails do not share one entire variable: $K_+^{\triangle}$ is an entire power series in y, but $K_-^{\triangle}$ is a power series in y^{-1} and has an essential singularity at $y=0$. The natural variable is the logarithmic character $v=\log y$, which is $r\ell_{\mathrm{t}}/2$ on the family. With the gauge $y\exp(-L_{\triangle}/8)$, the half-shifts (19) give

$$\begin{aligned}
\widehat{\Theta}_+(x_0^{\triangle},v)&=\sum_{n\ge 0}\exp\big((2n+1)v\big)\,\exp\Big(-L_{\triangle}\frac{(n+\frac12)^2}{2}\Big) &&=\widehat{\Theta}_-(x_0^{\triangle},-v),\\
\widehat{\Theta}_-(x_0^{\triangle},v)&=\sum_{n\ge 0}\exp\big(-(2n-1)v\big)\,\exp\Big(-L_{\triangle}\frac{(n-\frac12)^2}{2}\Big)-\exp(v-L_{\triangle}/8) &&=\widehat{\Theta}_+(x_0^{\triangle},-v),\\
\widehat{\Theta}_+(x_0^{\triangle},v)&=\exp(v-L_{\triangle}/8)\,K_+^{\triangle}(\exp v), \qquad \widehat{\Theta}_-(x_0^{\triangle},v)=\exp(v-L_{\triangle}/8)\,K_-^{\triangle}(\exp v).
\end{aligned}\tag{24}$$

Both are entire in v: the Gaussian decay dominates the linear character on every compact set. The lower rail is written on its own sheet from $n=0$, keeping its half-shift $n-\frac12$ and the one-step relation visible. The $n=0$ term of this lower-sheet presentation is the shared endpoint $\exp(v-L_{\triangle}/8)$, already present as the $n=0$ term of the upper rail, so it is subtracted once: the gauged image of the -1 overlap correction in (20). These are the global half-character rails of box (48) at the tower's scale $X=x_0^{\triangle}$: the two halves, under the reflection $\rho_{\triangle}(n)=-n-1$, of the half-character theta of H-PAP015 [54],

$$\widehat{\Theta}(X,v)=\sum_{n\in\mathbb{Z}}\exp\big((2n+1)v\big)\exp(-\pi X(n+\tfrac12)^2)$$

(owned by H-VER033 with H-AUD071-A, together with $\rho_{\triangle}$ and h; the split into the two halves is H-VER176's; the genealogy audit H-AUD083 retains the same pair). The reflection $\rho_{\triangle}$ is the triangular balance in bilateral form: it preserves T_+, reverses the odd index $h(n)=2n+1$, and its negative half is the lower sheet, $T_+(-m)=T_-(m)$. Their sum is the odd part of the half-index tower in (17),

$$B^{\alpha}_{r/2}(x_0^{\triangle}/4)=B^{\alpha}_{r}(x_0^{\triangle})+\widehat{\Theta}(x_0^{\triangle},r\ell_{\mathrm{t}}/2),\qquad \widehat{\Theta}=\widehat{\Theta}_++\widehat{\Theta}_-,\tag{25}$$

so the DVJ triangular family is, up to the factor $a_{\mathrm{t}}^{r/2}\exp(-L_{\triangle}/8)$, the global half-character theta at the tower's scale, and the reflection $r\mapsto -r$ is $v\mapsto -v$.

The sign flip $y_{\mathrm{t}}\mapsto -y_{\mathrm{t}}$, retained by H-PAP002 [52] in the parity partner $D_-^{\mathrm{dV}}(-y_{\mathrm{t}})$, is a separate involution: it does not exchange the triangular sheets. At index one it multiplies the j-th term of the half-index tower by $(-1)^j$, so its eigenspaces are the even and odd sublattices, and it commutes with the reflection. The two involutions generate the group $C_2\times C_2$ of H-VER044, whose third involution is their product, and there is no third independent one, so the parity sublattices, the sign flip and the reciprocal rails (one on each triangular sheet) stay three distinct notions. The specialization below takes $y_{\mathrm{t}}=+\sqrt{\alpha_{\mathrm{dVJ}}}$ only.

6.5 The three downstream DVJ specializations

Only *after* the generic family and odd-lattice identity have been established do the three audit specializations enter. Box (51) chooses the source Gaussian scale $L_0^{\triangle}=\log(2\pi)$; box (52) inserts the previously owned De Vries value α_{dVJ} [58, 52]; and box (53) specializes the character index, $r\longmapsto 1$. These are three coordinate specializations of an already established family, *not* three independent root equations. They are audit boxes off the certified path.

The index specialization acts on the family in its rail form (23), where r enters only through the rail variable $y = a_{\mathrm{t}}^{r/2}$. At index one that variable is $y_{\mathrm{t}} = \sqrt{a_{\mathrm{t}}}$; the family and its specialization, box (53), are

$$\begin{aligned} J_r^{\triangle} &= K_+^{\triangle}(a_{\mathrm{t}}^{r/2}) + K_-^{\triangle}(a_{\mathrm{t}}^{r/2}), \qquad r \in \mathbb{C}, \\ r \longmapsto 1: \qquad J_1^{\triangle} &= K_+^{\triangle}(y_{\mathrm{t}}) + K_-^{\triangle}(y_{\mathrm{t}}) = K_+^{\triangle}(y_{\mathrm{t}}) + y_{\mathrm{t}}^{-2} K_+^{\triangle}(y_{\mathrm{t}}^{-1}). \end{aligned} \tag{26}$$

The lower-sheet rail $K_-^{\triangle}(y_{\mathrm{t}})$ is the reflected negative lattice, kept beside the positive $K_+^{\triangle}(y_{\mathrm{t}})$. Consequently the frozen DVJ defect [59, 52] of box (54) *requires* the subtraction of the bracket $K_+^{\triangle}(y_{\mathrm{t}}^{-1}) - 1 = y_{\mathrm{t}}^2 K_-^{\triangle}(y_{\mathrm{t}}) - 1$: y_{t}^2 times the lower rail without its $n = 1$ cell ($j = -1$), whose weight is 1 since $T_-(1) = 0$. The registered Gaussian coefficient is $\Gamma_{\mathrm{src}}^{\mathrm{DVJ}}$, imported from H-PAP002 [52]; it is neither a new local normalization nor Euler's Gamma function. Removing the bracket would replace the canonical unilateral closure with an inequivalent bilateral scalar equation.

H-PAP002 [52] and H-VER089 retain their original two-root information. At the smaller De Vries root the frozen defect vanishes as in box (54); that is a downstream recognition of the previously selected solution, not a new proof of root selection from the whole lattice. This specialization preserves the negative tail rather than suppressing it.

6.6 The DVJ tower inside the theta lattice

Sections 6.4 and 6.5 compose into one walk from the theta lattice to the DVJ defect, each step an equation above.

1. The start is the half-index theta lattice $B_{r/2}^{\alpha}(x_0^{\triangle}/4)$: one bilateral Gaussian lattice over all $j \in \mathbb{Z}$, with no sheet and no rail chosen.

2. Its parity split (17) separates the even indices, which are the tower $B_r^{\alpha}(x_0^{\triangle})$ again, from the odd ones.

3. The odd indices fall on the two sheets (18) at the opposite half-shifts (19), and taking out the common factor leaves the triangular family as the two sheets, (20).

4. The odd part is, up to the factor $a_{\mathrm{t}}^{r/2} \exp(-L_{\triangle}/8)$, the same tower at a character half a quasi-period away. On the two sheets the triangular exponents are the tower's Gaussian exponent with a linear half-step of opposite signs, $L_{\triangle} T_+(n) = \pi x_0^{\triangle} n^2 + L_{\triangle} n/2$ and $L_{\triangle} T_-(n) = \pi x_0^{\triangle} n^2 - L_{\triangle} n/2$, and the half-step is absorbed by the character:

$$\begin{aligned} J_r^{\triangle} &= B^{\alpha}_{r - L_{\triangle}/(2\ell_{\mathrm{t}})}(x_0^{\triangle}) \qquad (a_{\mathrm{t}} \neq 1), \\ B_{r/2}^{\alpha}(x_0^{\triangle}/4) &= B_r^{\alpha}(x_0^{\triangle}) + a_{\mathrm{t}}^{r/2} \exp(-L_{\triangle}/8)\, B^{\alpha}_{r - L_{\triangle}/(2\ell_{\mathrm{t}})}(x_0^{\triangle}). \end{aligned} \tag{27}$$

   The shift $L_{\triangle}/(2\ell_{\mathrm{t}}) = \pi x_0^{\triangle}/\ell_{\mathrm{t}}$ is half the quasi-period of the tower in its index. This is the classical translation of the third Jacobi theta function by a half quasi-period into the second [25]; H-VER176 proves it termwise, and in nome form it needs no condition on the base. A whole quasi-period only returns the tower times an elementary factor [25] and never reaches the odd sublattice.

5. The two sheets carry the two rails (23), reciprocal by the balance of the sheets, with $J_r^{\triangle} = K_+^{\triangle}(a_{\mathrm{t}}^{r/2}) + K_-^{\triangle}(a_{\mathrm{t}}^{r/2})$; in the logarithmic character they are the global half-character rails $\widehat{\Theta}_+$ and $\widehat{\Theta}_-$ at $X = x_0^{\triangle}$, entire in v (24), and their sum is the half-character theta $\widehat{\Theta}(x_0^{\triangle}, r\ell_{\mathrm{t}}/2)$ (25).

6. At index one, (26), the rail variable is y_{t}, and in the DVJ source defect of box (54) the lower rail enters through the bracket $y_{\mathrm{t}}^2 K_-^{\triangle}(y_{\mathrm{t}}) - 1$ built from it, which the defect must subtract.

The DVJ family is therefore not appended to the theta lattice: it is the lattice's own odd sublattice, which is, up to the factor $a_{\mathrm{t}}^{r/2}\exp(-L_{\triangle}/8)$, the tower half a quasi-period away, and both of its rails, the lower one included, reach the DVJ defect.

The arithmetic stages come with it. The stages of the Euler-log and prime-theta layers above hold at every complex index (H-VER175), and by (27) the family is the tower at the index $r' = r - L_{\triangle}/(2\ell_{\mathrm{t}})$. There the stage cells are $(\delta_n^{\times} g^{\alpha})_{r'}(x_0^{\triangle}) = a_{\mathrm{t}}^{rn}\exp(-L_{\triangle}T_+(n)) + a_{\mathrm{t}}^{-rn}\exp(-L_{\triangle}T_-(n))$, one cell of each sheet, because $L_{\triangle}(n^2+n)/2 = L_{\triangle}T_+(n)$ and $L_{\triangle}(n^2-n)/2 = L_{\triangle}T_-(n)$. Collecting the cells with each stage's coefficients gives box (49) (H-VER176):

$$
\begin{aligned}
J_r^{\triangle} - 1 = (\exp_{\star}(\mathcal{L}_{\times})\, g^{\alpha})_{r'}(x_0^{\triangle}) &= \sum_{n\geq 1}\Big[a_{\mathrm{t}}^{rn}\exp(-L_{\triangle}T_+(n)) + a_{\mathrm{t}}^{-rn}\exp(-L_{\triangle}T_-(n))\Big],\\
(\mathcal{L}_{\times}\, g^{\alpha})_{r'}(x_0^{\triangle}) &= \sum_{p}\sum_{a\geq 1}\frac{1}{a}\Big[a_{\mathrm{t}}^{rp^a}\exp(-L_{\triangle}T_+(p^a)) + a_{\mathrm{t}}^{-rp^a}\exp(-L_{\triangle}T_-(p^a))\Big],\\
(\mathcal{P}_{\times}\, g^{\alpha})_{r'}(x_0^{\triangle}) &= \sum_{p}\Big[a_{\mathrm{t}}^{rp}\exp(-L_{\triangle}T_+(p)) + a_{\mathrm{t}}^{-rp}\exp(-L_{\triangle}T_-(p))\Big].
\end{aligned}
\tag{28}
$$

On the family the full line, the Euler-log stage and the prime stage are the two triangular sheets summed over $n \geq 1$, over prime powers and over primes. No stage drops a rail: the prime stage keeps the lower sheet's prime cells beside the upper sheet's. In the first line the literal sheet form (20) is visible: its two $n = 0$ cells, less the overlap correction, are the 1 on the left, the zero cell of the tower at r'.

6.7 The distinct zero-character recognition

The setting $r = 0$ is the separate recognition branch box (55). There the generic bilateral tower agrees with $\Theta_0^{\mathrm{HG}}(x)$, independently of a_{t}. The triangular zero-character member, $J_0^{\triangle} = 2\sum_{n\geq 0}\exp(-L_{\triangle}T_+(n))$, the two sheets at equal weight, is a half-shifted odd-lattice expression: by (27) it is the tower at the nonzero index $-L_{\triangle}/(2\ell_{\mathrm{t}})$, and it is *not* identified with that untwisted theta function. Neither the zero-character recognition nor the corresponding H-PAP005 [57] Ξ readout is an input to the index-one DVJ specialization.

The results of this section remain the scalar-source theorems owned by H-VER175 (entire tower and its analytic realizations) and H-VER176 (triangular family and negative-tail-preserving insertion). They do not supply the missing retained FULL40 transport of H-VER172, a new zero-preserving DVJ-to-Xi intertwiner, or a Riemann-hypothesis conclusion. Those boundaries are stated in box (57) and Appendix D.

7 The two-leg conductor representation

7.1 Two independent bilateral legs

The preceding sections realize the full-line arithmetic on one bilateral Gaussian-character tower. We now take *two independent* legs, with positive real heat parameters x_1, x_2 and complex character indices r_1, r_2. Both use the same real $\ell_t = \log a_t$, but neither is specialized to the frozen DVJ root or to the index value one. The two cells $c^{(1)}$ and $c^{(2)}$ are defined in box (50). Their product retains all four sign choices of a pair of nonzero lattice indices:

$$c^{(1)}(d)c^{(2)}(e) = \sum_{s,t\in\{-1,1\}} \exp\bigl(-\pi(x_1 d^2 + x_2 e^2) + \ell_t(s r_1 d + t r_2 e)\bigr), \qquad d, e \geq 1. \tag{29}$$

Thus no modulus, sign choice or one-sided restriction is inserted when the cell pair is formed. The positive conductor of the corresponding signed lattice pair is the absolute product of its two nonzero indices. In particular, the zero cells of the complete bilateral tower are *not* conductor-positive factor pairs; this assertion does not remove the zero cell from the underlying tower.

For each conductor $n \geq 1$, define its scalar packet

$$C_n^{\otimes} := \sum_{de=n} c^{(1)}(d)c^{(2)}(e),$$

the *finite* Dirichlet convolution [3, 6] of the two cell sequences, as registered in box (50). The divisors d, e with $de = n$ run over the absolute values of the signed pairs; (29) supplies their four signed amplitudes. This is the scalar, charge-forgetting realization of the signed-pair enumeration in the closed one-leg conductor intertwiner and substitution theorem of H-VER173-A: the theorem acts on the signed-pair route source, on all sixteen of its signed-index routes, with a relabelling of the charges. The two leg functions are supplied by H-VER175. The equality is an identity of scalar coefficients, not a construction of charged route packets.

7.2 One-leg dilations and exact inclusion–exclusion

Let p be a *prime*. Denote its two one-leg dilations by $\delta_p^{\times,(1)}$ and $\delta_p^{\times,(2)}$: they are the separate substitutions

$$(x_1, r_1) \longmapsto (p^2 x_1, p r_1), \qquad (x_2, r_2) \longmapsto (p^2 x_2, p r_2),$$

with the other leg held fixed, as registered in box (50). Direct substitution into each Gaussian cell gives

$$\begin{aligned} (\delta_p^{\times,(1)} c^{(1)})(d) &= c^{(1)}(pd), \\ (\delta_p^{\times,(2)} c^{(2)})(e) &= c^{(2)}(pe). \end{aligned} \tag{30}$$

The notation on the left of (30) means substitution in that cell's own pair of parameters; the other leg is held fixed. Consequently, the first one-leg image enumerates exactly the factor pairs of conductor pn whose *first* factor is divisible by p, while the second image enumerates those whose *second* factor is divisible by p.

Because p is prime, a factorization $de = pn$ must have $p \mid d$ or $p \mid e$. The multiplicities of the two subsets obey the pointwise inclusion–exclusion identity

$$\begin{aligned}[p \mid de] &= [p \mid d] + [p \mid e] - [p \mid d][p \mid e], \\ [p \mid d][p \mid e] &= [p \mid \gcd(d, e)].\end{aligned} \tag{31}$$

Here the square brackets denote 0–1 divisibility indicators, as in the registered two-leg box. The second line of (31) identifies the *overlap* with the common prime divisor, rather than merely with two independently selected heat operations. The prime hypothesis matters: the first line does not hold for an arbitrary composite multiplier.

When both factors are divisible by p, write them as pd and pe. Their product is $p^2de = pn$, hence $de = n/p$; such a pair exists only if $p \mid n$. In the packet notation, the overlap is exactly

$$\begin{aligned}&\sum_{\substack{de=pn \\ p \mid \gcd(d,e)}} c^{(1)}(d)c^{(2)}(e) \\ &\qquad = [p \mid n]\, \delta_p^{\times,(1)} \delta_p^{\times,(2)} C^{\otimes}_{n/p}.\end{aligned} \tag{32}$$

The right side of (32) is interpreted as zero when $p \nmid n$: there is then no $C^{\otimes}_{n/p}$ term to evaluate. Combining the two one-leg images from (30) with the intersection (32) proves the precise one-leg conductor identity displayed in box (50). No limit, Euler product or analytic readout participates in this finite, coefficientwise argument. For completely multiplicative legs this one-leg law is the one-prime case of the Busche–Ramanujan identity for specially multiplicative functions [32, 33], and it has the shape of the Hecke relation for the operators T_pT_n [34, 35]. The proof above does not need the cell sequences to be multiplicative, and no Hecke operator is used.

The subtraction is necessary, not a normalization convention. For $p = n = 2$, the packet of conductor 4 contains the factor pair $(2, 2)$ in *both* one-leg images. It must occur once, so the shared term $\delta_2^{\times,(1)}\delta_2^{\times,(2)}C_1^{\otimes}$ is subtracted. The corresponding hostile test in `test_dvj_xi_bridge_witness4.py` detects exactly the missing-intersection error.

7.3 Common heat is not one-leg conductor motion

Simultaneously applying the two leg dilations scales the two signed indices by p. Their conductor therefore changes by p^2, *not* by p. The final line of box (50) is the exact statement: the common-heat image is the part of conductor p^2n whose *two* absolute factors are divisible by p. It is generally a *proper part* of the full $C^{\otimes}_{p^2n}$; factor pairs having only one p-divisible factor are omitted. Conversely, (32) involves the common heat on conductor n/p solely as an intersection correction inside the *one-leg* $n \mapsto pn$ law.

Each individual packet is a finite divisor sum. Where one sums these packets over all conductors, the Gaussian absolute majorants for the two independent legs, as in Lemma 5.1, give locally uniform convergence on compact character-index sets for positive heat parameters. This analytic control is distinct from the finite combinatorial proof above, and it does not justify rearranging the uncollapsed alternating logarithm of Section 3.

7.4 Comparison with the charged two-leg law, and the retained boundary

This subsection is a comparison with work in progress, not a result of this paper. It sets the scalar law beside the charged, route-resolved forty-label transport that H-VER173 still leaves open (its gate G3); that transport is to be taken up in future work or a companion paper. Nothing here is a premise of any result in this paper.

H-VER173-A proves the closed one-leg conductor intertwiner and substitution theorem on the signed-pair route source, with a charge relabel as well. The scalar law here is a **MATCH** to that relation: its signed pairs, amplitudes, one-leg images and overlap are those of the owner's scalar witness. It is not a claim that the scalar packet itself retains the owner's charge vectors, the sixteen route placements, or the forty source labels.

At the charge-only comparison layer, dilation acts by the pullback $m \mapsto pm \pmod 8$. For odd p this is a permutation of the eight charge classes and is a **MATCH** to the corresponding H-VER173-A relabel. For $p = 2$ it is not invertible on that finite shadow; the signed-lattice one-leg law remains valid, but the finite shadow loses information. Accordingly, any identification of the resulting *charged*, *route-resolved* two-leg object with the complete FULL40 conductor action is **NOT YET IDENTIFIED**: that is the separate open H-VER173 G3 obligation, not an automatic consequence of the scalar match.

The bridge-audit witness checks the three scalar claims (Dirichlet convolution, one-leg inclusion–exclusion and the restricted common-heat image) at 200 decimal digits with complex character indices; its tests include both missing-intersection and whole-packet/common-heat hostile controls. The witness is `witness4_two_leg_conductor.py`. These numerical receipts corroborate, rather than replace, the finite exact factor-pair proof. The scalar theorem owner is H-VER173-A, together with H-VER175 for the two legs; this section creates no new ingress into H-VER172 or the certified path. The unproved FULL40 transport remains explicit in box (57) and Appendix D.

8 Readout comparison, behind the firewall

8.1 The readout is terminal, not an arithmetic premise

Sections 2–4 prove their identities in the formal multiplicative dilation algebra; Section 5 then supplies the separate bilateral Gaussian-character realization. Neither proof uses a Dirichlet series, an Euler-product value, or analytic continuation of the Riemann zeta function. Only after those results have been established do we consider the classical substitution recorded in the *firewall box* (56).

For a complex argument s with $\operatorname{Re} s > 1$, send each basis dilation $\delta_n^{\times}$ to the scalar n^{-s}. The multiplicative rule is respected on basis elements:

$$\delta_m^{\times}\delta_n^{\times} \longmapsto (mn)^{-s} = m^{-s}n^{-s}, \qquad \operatorname{Re} s > 1. \tag{33}$$

The equality in (33) follows from box (37); it says nothing about evaluating an arbitrary *formal* infinite series. Scalar evaluation of an infinite coefficient sum additionally requires convergence on the specified domain. In particular, the coefficientwise-finite logarithm of box (39) is not, merely for that reason, an absolutely convergent uncollected series at any fixed s.

8.2 The two classical Dirichlet-series comparisons

The full-line element $\mathcal{N}_\times$ itself reads out as the classical zeta Dirichlet series on this domain [3, 5]. The first stage is the Euler-log element $\mathcal{L}_\times$, whose prime-power coefficients were proved in Section 3. After collecting those coefficients, the readout in box (56) is the familiar $\log\zeta(s)$ for $\operatorname{Re} s > 1$ [5]. The domain restriction is substantive: the positive-integer Dirichlet series and the associated Euler product converge absolutely there, so their logarithms may be compared using the branch fixed by the convergent Euler-product expansion. This is a classical identification [4, 3], not a novel proof of an Euler product. The literature contains stronger analytic prime/zero bridges, including Suzuki's screw function, a prime-power sum plus explicit archimedean terms whose one-sided Fourier transform gives the logarithmic derivative of the completed zeta function up to an explicit elementary factor, and operator/trace constructions of Meyer, Connes, and Connes–Consani–Moscovici [44, 45, 46, 47]. Those are comparisons at or beyond the scalar/spectral readout boundary; they are not used to reconstruct this paper's source object.

For the second stage, the power substitution of box (41) acts on readout indices by

$$\psi_k^\times(\delta_n^\times) = \delta_{n^k}^\times \longmapsto n^{-ks}, \qquad k, n \geq 1, \quad \operatorname{Re} s > 1. \tag{34}$$

Under (34), the algebraic inversion in (8) becomes an ordinary perfect-power Möbius inversion of the *already evaluated* Euler logarithm. In particular, the second readout of box (56) can be written as

$$\mathcal{P}_\times \longmapsto \sum_p p^{-s} = \sum_{k\geq 1} \frac{\mu_{\mathrm{M}}(k)}{k} \log\zeta(ks), \qquad \operatorname{Re} s > 1. \tag{35}$$

The series on the left of (35) is the classical *prime zeta function*, and (35) is the prime zeta identity, which goes back to Glaisher and was studied by Landau and Walfisz and by Fröberg [16, 17, 18]. It is the scalar readout of $\mathcal{P}_\times$, not a second name for the formal prime-supported element itself.

The interchanges used in these two *collected* readouts are justified on the indicated half-plane by elementary absolute bounds. The first uses the $1/a$ weights; the second also uses the exponent-divisor majorant of (14):

$$\begin{aligned} \sum_p \sum_{a\geq 1} \frac{p^{-a\operatorname{Re} s}}{a} &\leq \sum_{n\geq 2} n^{-\operatorname{Re} s} < \infty, \\ \sum_p \sum_{a\geq 1} \frac{p^{-a\operatorname{Re} s}}{a} \sum_{k\mid a} |\mu_{\mathrm{M}}(k)| &\leq \sum_{n\geq 2} n^{-\operatorname{Re} s}, \qquad \operatorname{Re} s > 1. \end{aligned} \tag{36}$$

The second line of (36) bounds the absolute values of the Möbius-weighted readout terms *before* their cancellation. The argument relies on unique prime-power indices and on the coefficient-one inequality in (14). It does *not* replace the coefficientwise proof in Sections 3 and 4, nor does it retroactively justify arbitrary rearrangements of the uncollected alternating ordered-factorization series.

8.3 What the scalar readout forgets

At any *fixed* s, the readout of an infinite family of basis coefficients is only one scalar value. It no longer specifies the joint action $(x, r) \mapsto (n^2 x, nr)$ of box (43); nor does it retain the two signed character rails, their zero-cell bookkeeping, or the two independently dilated legs and conductor

overlaps of box (50). The source representation was specified before the readout; one must not reconstruct it by declaring an inverse to this scalar substitution.

Part of this loss can be located at the step where the two reciprocal rails are summed. Keep the rails and orientations ordered, as the positive half of the half-index lattice. In the $C_2 \times C_2$ character basis of H-VER044 for the sign flip $y_{\mathrm{t}} \mapsto -y_{\mathrm{t}}$ and the reflection of Section 6, that half has four components: the even tower, the difference of its two orientations, the odd (triangular DVJ) part $\widehat{\Theta} = \widehat{\Theta}_+ + \widehat{\Theta}_-$ at $X = x_0^{\triangle}$, and the difference $\widehat{\Theta}_+ - \widehat{\Theta}_-$ of its two reciprocal rails, the upper-sheet rail on T_+ and the lower-sheet rail on T_- (24). Summing over the reflection, which gives the full bilateral lattice, keeps the two reflection-even components; the two reflection-odd ones, the orientation difference and the rail difference, are its kernel as a map on ordered pairs. They are lost once the character index is fixed, so that each component is a single value. As functions of the logarithmic character v they are not lost: the positive orientation and the upper rail carry the positive powers of $\exp v$, the negative orientation and the lower rail the negative powers, and the Laurent split of the sum returns each half. That is the precise loss at the reflection-summing step at a fixed index; the later scalar readout may lose more, so the two components are not an exhaustive list of what the readout forgets.

This point is deliberately narrower than a blanket claim of non-uniqueness for Dirichlet series. Knowing the *complete analytic function of s on a half-plane* under the usual absolute convergence hypotheses can determine its Dirichlet coefficients [3]. That coefficient-uniqueness observation does not, without separately proved additional structure, recover the original bilateral character variables, signed route data or forty source labels. Owner ruling M1, the paper's loss-of-information firewall, calls the readout *lossy* in exactly this proof-theoretic sense: it cannot be promoted to an upstream source or an inverse transport theorem.

Nor should the formal substitution $\delta_n^{\times} \mapsto n^{-s}$ be conflated with a Mellin integral of the theta kernel. A Mellin readout needs its own kernel, normalization, integration domain and interchange proof; in particular any *Euler gamma function* appearing in that separate classical calculation must not be confused with the imported DVJ scalar $\Gamma_{\mathrm{src}}^{\mathrm{DVJ}}$ of H-PAP002 [52]. No such transform is needed for the equations of this section.

8.4 No reverse ingress or analytic-continuation claim

All conclusions above are classical, half-plane-restricted **READOUT** comparisons: the firewall of box (56) has no outgoing load-bearing edge. The analytic entireness established by H-VER175 is entireness in the *character index* r at positive Gaussian scale, not a proof of continuation of $\log \zeta(s)$ or the prime-zeta readout beyond $\operatorname{Re} s > 1$. Indeed the prime zeta function cannot be continued beyond $\operatorname{Re} s > 0$: that line is its natural boundary [17]. A statement about continuation, zeta zeros or their relation to a frozen DVJ root would require a separately typed argument. H-PAP016 [55] (Paper A) is the independent DVJ-to-Ξ readout-bridge comparison; it is not imported as a premise here.

In particular, no scalar readout proves the unconstructed charged, route-resolved FULL40 lift. H-VER172 remains the open future consumer named in box (57); Appendix D retains the complete ledger of non-claims.

9 In the literature

The ingredients of this paper are classical, and the arithmetic at its core is older than the zeta function's analytic theory. The point of this section is therefore to state the novelty boundary conservatively: what is credited to the literature, and what the paper adds to it.

The formal algebra. Formal series $\sum_n b(n)\delta_n^{\times}$ multiplied by Dirichlet convolution, box (37), form the ring of arithmetic functions [3, 6, 27]; Cashwell and Everett showed that this ring is a unique factorization domain, isomorphic to formal power series in countably many indeterminates [9]. Reading the index n as a dilation is also established: Hedenmalm, Lindqvist and Seip identify Dirichlet series with systems of dilated functions $\varphi(nx)$ and, through Bohr's lift, with functions of one variable per prime [23], and Bost and Connes realize the multiplicative semigroup of the positive integers by isometries in a Hecke algebra whose partition function is ζ [22]. Nothing in Section 2 is new mathematics; only the names (full line, augmentation) are the paper's.

Primitive extraction beyond number theory. There is a broader algebraic precedent for taking a convolution logarithm in order to expose primitive structure. In a connected filtered cocommutative bialgebra over a field of characteristic 0 the Eulerian idempotent is the convolution logarithm of the identity and projects onto the primitive elements [36]. This is a structural analogy, not an identification with the arithmetic-function algebra used here. Big Witt vectors provide another closely related product/logarithm pattern: a formal Euler product has ghost coordinates obtained from a logarithmic derivative, while necklace-ring descriptions encode primitive-versus-repeated cyclic data through Möbius inversion [37, 38, 39]. These constructions motivate the language of primitive extraction, but the coefficients and source action in this paper are those proved in H-VER174 rather than imported from Witt or Hopf algebra theory.

The first stage is Linnik's identity. The logarithm over ordered factorizations, box (39), with its alternating harmonic weights and its cutoff $2^k \le n$, equals the coefficient of n^{-s} in $\log\zeta(s)$: $1/a$ at a prime power p^a and zero otherwise, that is, the von Mangoldt function divided by $\log n$. This is Linnik's identity [10, 11]; it is stated in exactly this form, with the same cutoff, by Sprittulla [12]. The counts of ordered factorizations go back to Kalmár and Hille [13, 14], and at a prime power they are MacMahon's compositions of the exponent [15]. The Euler logarithm it equals, box (40), is the logarithm of the Euler product [1, 4, 5]. What this paper adds is only the reading order: the sum is taken over the full positive-integer line, no prime is singled out, and the cancellation of every integer with two or more distinct prime factors is read as an output.

The second stage and the inverse are Riemann's. Riemann's prime-power counting function gives weight $1/a$ to p^a, exactly the coefficients of the Euler logarithm, and he recovers the primes from it by Möbius inversion over perfect powers [1, 2]. So the hierarchy full line, prime powers, primes is Riemann's, and box (41) is its formal coefficient version. At the level of Dirichlet series it is the prime zeta identity

$$\sum_p p^{-s} = \sum_{k\ge 1} \frac{\mu_{\mathrm{M}}(k)}{k} \log\zeta(ks), \qquad \operatorname{Re} s > 1,$$

going back to Glaisher and studied by Landau and Walfisz and by Fröberg [16, 17, 18]; the two stages have been composed into one formula for the prime-counting function in an informal note by McKenzie [19]. The Möbius divisor sum and the count of square-free divisors used in Sections 4 and 5 are standard [6, 7]. That the convolution exponential returns the full line, box (42), is the integer case of Beurling's relation between generalized integers and generalized primes [20, 21].

Primitive objects and repeated windings. Dynamical zeta functions give a second strong analogy. Their Euler products are indexed by primitive periodic or closed orbits, while the logarithm generates repeated traversals of each primitive orbit with the familiar reciprocal repetition weights [40]. Deninger's arithmetic-dynamical programme makes the corresponding prime/closed-orbit analogy explicit in the study of zeta functions [41]. In this paper the analogous pair is a prime and its powers p^a: the analogy explains the terminology of primitive support and repeated winding, but no dynamical system is assumed or constructed.

Multiplicative monoids, cyclotomy and endomorphisms. The Bost–Connes system already realizes the multiplicative semigroup of positive integers by endomorphism-like operators in a cyclotomic setting [22]. Marcolli's endomotive treatment makes this comparison particularly close: roots of unity carry compatible actions of the multiplicative semigroup of positive integers [42]. More recently, Gotti and Krause extend Euler products from $\mathbb{N}^\times$ to Krull monoids with torsion class group equipped with a scale: the product over the strong atoms gives the zeta function of the submonoid they generate, which is the whole monoid only when the monoid is factorial [43]. These are source-side precedents for multiplicative-semigroup organization; they do not supply the H-VER040 dilation action, the two-stage projector, or the bilateral DVJ realization proved here.

The tower is a Jacobi theta function. The bilateral tower of box (43) is Jacobi's third theta function at imaginary argument with a character, and its seed is the paired term of that sum [24]. The relation $\partial_r^2 B^\alpha = -(\ell_t^2/\pi)\,\partial_x B^\alpha$ in Section 6 is the heat equation of the theta functions written in the tower's variables [26]. The triangular family of box (47) is Ramanujan's two-variable theta function $f(A,B)$ evaluated on the tower's character, with the two triangular sheets as its exponents, $f(A,B) = \sum_{n\in\mathbb{Z}} A^{T_+(n)} B^{T_-(n)}$; its zero-index member, the two sheets at equal weight, is twice Ramanujan's series over the triangular numbers, and its reflection under a change of sign of the index follows from Ramanujan's Entry 18 [28]; its product form is Ramanujan's Entry 19, Jacobi's triple product [28], in which Section 6.4 splits off the lower product's $n = 0$ factor. The odd-lattice extraction is the even and odd split that produces the second Jacobi theta function [24], and the identification of the odd part with the tower half a quasi-period away, up to an elementary factor, is the classical translation of the third theta function by a half quasi-period [25]. The half-character theta and its reflection belong to the project's own source (H-PAP015 [54]); its split into the two rails is this paper's notation, not a priority claim. We found no earlier source for the Euler-log theta and the prime theta, boxes (45) and (46), as coefficient actions on one bilateral theta lattice under the joint dilation of scale and index. They are termwise substitutions, however, and no priority is claimed for them.

Analytic and spectral prime–zero bridges. Several exact or approximate constructions connect prime-power data to zeta or Xi objects, but at a different information boundary from this paper. Suzuki's screw function is a prime-power sum plus explicit archimedean terms, and its one-

sided Fourier transform gives the logarithmic derivative of the completed zeta function up to an explicit elementary factor [44]; Meyer and Connes obtain explicit-formula or spectral realizations in which prime powers and zeta zeros appear in a common operator/trace framework [45, 46]. The Connes–Consani–Moscovici zeta-spectral-triple programme keeps both reciprocal prime-power directions in its local explicit-formula terms and constructs finite prime-generated spectral operators and regularized determinants [47]. Its limiting identification with the entire Xi function remains open. The Euler–Hadamard hybrid of Gonek, Hughes and Keating gives a further prime/zero comparison, but only in an approximate regime [48]. Tate's adelic construction supplies the classical global theta–Mellin–Euler precedent [49, 50]; in the present paper Mellin and scalar zeta are terminal comparisons, not source ingress. These works therefore provide comparison targets, not premises for the bilateral source construction below.

Analysis and its limits. Lemma 5.1 uses only a majorant, Tonelli's theorem for non-negative series and Weierstrass's theorem on locally uniform limits of holomorphic functions [31]. Its entireness is in the index r alone. At fixed r the prime theta is a power series in $w = \exp(-\pi x)$ whose exponents p^2 have gaps, and by Fabry's gap theorem it has the unit circle $|w| = 1$ as a natural boundary [29]; likewise the prime zeta readout has the line $\operatorname{Re} s = 0$ as a natural boundary [17]. Both facts agree with the paper's non-claims: no continuation in the scale or in the readout variable is asserted.

The two-leg law is a Busche–Ramanujan identity. For completely multiplicative legs, the one-leg conductor law of box (50) is the one-prime case of the Busche–Ramanujan identity for specially multiplicative functions [32, 33], and it has the shape of the Hecke relation $T_p T_n = T_{pn} + [p \mid n]\, p^{k-1} T_{n/p}$ [34, 35]. The paper's statement is the finite inclusion–exclusion over the factor pairs of a Dirichlet convolution of two independent theta legs, proved directly in Section 7 for cell sequences that need not be multiplicative; no Hecke operator is used.

Project sources. The frozen De Vries data, the source defect and its Gaussian scalar come from the project's De Vries–Jordan–Jacobi (DVJ) work [58, 59, 52]; the zero-index tower is the full-lattice theta source of the companion paper on the entire Xi carrier [57]; the ordinary constants π and exp enter through their owner paper [51]; and the forty-label source on which the open aim is stated is that of [54].

What is claimed. No priority is claimed for the Euler product or its logarithm, for Linnik's identity, for Möbius inversion over perfect powers or the prime zeta identity, for Beurling's relation, for the theta relations, or for the Busche–Ramanujan form of the two-leg law; they are cited above. The paper's contribution is the source order, full line first and the primes as an output; the typing of the two stages in the dilation algebra and the ownership of each step; the realization of both stages on one bilateral Gaussian character lattice under a joint dilation of scale and index, entire in the index; and the triangular insertion of the DVJ defect with the negative tail of the lattice, the lower sheet T_- on its own rail, kept.

10 Conclusion

This paper began from the full multiplicative line, not from the primes. Its source is the complete positive-integer element box (38): every positive integer, each with coefficient one, and no prime subset, primality test or prime mask in the definition.

Starting from all positive integers, the convolution logarithm of Section 3 removes, exactly, every integer with two or more distinct prime factors, and leaves the prime-power windings $p, p^2, p^3, \dots$ with their reciprocal weights $1/a$, as in box (40). The perfect-power Möbius inversion of Section 4 then removes the repeated windings and leaves the primitive support, the primes, as in box (41), while the convolution exponential returns the full line, box (42). There are two eliminations, not one:

$$\text{integers with two or more distinct prime factors} \longmapsto 0, \qquad p^a,\ a>1 \longmapsto 0,$$

the first by the logarithm and the second by the Möbius stage. Only after both does the prime support remain. The primes are therefore a consequence of the source algebra, not a premise imposed on it. The source and both operators are defined without naming a prime: the logarithm over ordered factorizations, and the Möbius weights as the coefficients of $\mathcal{N}_\times^{-1}$. Unique factorization enters only where the output is recognized as the primes, as it must; a prime-free source and prime-free operators are not a prime-free proof of that recognition, and the paper claims only the former.

This changes the emphasis of the Euler product without correcting it: $\prod_p (1-p^{-s})^{-1}$ of course equals $\zeta(s)$ on $\operatorname{Re} s > 1$. The question is which object comes first. Here the source is the sum over all positive integers, and the prime-indexed product is what remains once the contributions of the integers with several prime factors have cancelled and the repeated prime-power windings have been separated from their primitive generators. The non-primes need not be excluded at the beginning: they may be included from the outset and left to cancel by the multiplicative algebra itself.

The arithmetic cancellation alone is not the result of the paper. It is classical, and Section 9 credits it: the first stage is Linnik's identity [10, 12] and the second is Riemann's inversion [1, 2]. What the paper adds is the realization of both stages on the bilateral DVJ source without replacing that source by a lossy scalar readout, and that meant inserting one infinite construction into another. The bilateral DVJ alpha tower of Section 6 is infinite and self-recursive: its paired lattice modes obey the two-step reciprocal recurrence (15) and share one second-order heat relation (16), and the De Vries data whose Jordan structure has order three [59, 52] enter it at its specializations. The infinitely many arithmetic dilations act on that infinite lattice jointly in scale and index. Lemma 5.1 makes the collected stages of this infinite-in-infinite substitution converge, entire in the character index, and Section 6 carries the full-line elimination into the DVJ family, (28): at the index moved by half a quasi-period, the full line, the Euler-log stage and the prime stage are the two triangular sheets summed over $n \geq 1$, over prime powers and over primes. Every stage keeps both reciprocal rails of the odd sublattice: $K_+^\triangle$ on the upper triangular sheet T_+ and $K_-^\triangle$ on the lower sheet T_-, the negative tail.

The bilateral carrier has the same shape at three levels. The two triangular sheets T_+ and T_- carry the two DVJ rails $K_+^\triangle$ and $K_-^\triangle$, and these are, up to the gauge $\exp(v - L_\triangle/8)$, the two global half-character rails $\widehat{\Theta}_+$ and $\widehat{\Theta}_-$ at the tower's scale, box (48). At each level the same reflection $\rho_\triangle$ appears in the corresponding coordinates: as the balance $T_-(n) = T_+(n-1)$ on the sheets, as the reciprocity $K_-^\triangle(y) = y^{-2} K_+^\triangle(y^{-1})$ on the DVJ rails, and as $v \mapsto -v$, $\widehat{\Theta}_-(X, v) = \widehat{\Theta}_+(X, -v)$, on the half-character rails. At each level the two halves, written literally from $n = 0$, share one

endpoint, which is subtracted once as its own term: the -1 of (20) and of (23), and its gauged image $\exp(v - L_\triangle/8)$ in (24). In Jacobi's triple product the lower product's $n = 0$ factor, $1 + a_{\mathrm{t}}^{-r}$, likewise stands apart, and what remains is symmetric under $r \mapsto -r$ (21). The negative tail is therefore not a correction added to a one-sided object: it is the second half of a reflected pair, and the one endpoint the two halves share is kept explicit at every level.

The boundary is therefore sharp. The prime extraction is arithmetic and does not require the DVJ source. The paired reciprocal rails are what realize the extracted object bilaterally: they do not create the prime projector, they carry it without collapsing its reciprocal structure. The reflection that exchanges the reciprocal rails and the independent sign flip $y_{\mathrm{t}} \mapsto -y_{\mathrm{t}}$ on the half-index lattice, whose eigenspaces are the even and odd sublattices, are two commuting involutions, the $C_2 \times C_2$ of H-VER044 (Section 6), and the rails keep their orientation data until they are summed. At a fixed index the sum erases exactly their reflection-odd part; as functions of the logarithmic character v the Laurent split still separates them (Section 8).

Neither the multiplicative logarithm, Möbius inversion, primitive extraction, nor the algebra of repeated windings is new in isolation. Closely analogous structures occur in Euler products and the prime zeta inversion, in Eulerian idempotents, in Witt-vector and necklace constructions, in dynamical zeta functions with their primitive closed orbits, and in multiplicative-semigroup systems (Section 9). The innovation lies in their composition. In this programme it was putting these analogous structures together, with the DVJ tower set inside the theta lattice, that brought the mechanism into view. Section 6.6 walks through that setting step by step: the DVJ family is the odd sublattice of the half-index theta lattice, which is, up to the factor $a_{\mathrm{t}}^{r/2} \exp(-L_\triangle/8)$, the tower itself half a quasi-period away, and both of its rails reach the DVJ defect. From the conventional scalar Euler product the full-line cancellation is easy to read past, because the primes are written into the product from the start.

In this sense the paper replaces "start with the primes" by "start with everything and derive the primes".

The result is established on the source-scalar side: by H-VER174, H-VER175, H-VER176 and H-VER173-A, on the H-VER040 dilation monoid. It does not claim the retained FULL40 transport. The lift of these stages to all forty labelled source components is the open aim box (57), for H-VER172, and the scalar theorem should not be mistaken for that stronger retained statement. Nor does the paper claim a continuation of its readouts, a distinguished-root-to-zero correspondence, or the Riemann hypothesis; the ledger of these boundaries is Appendix D.

What has been established is narrower, and clean: on the full multiplicative source, the prime support is obtained by exact cancellation and primitive extraction, and this arithmetic admits a bilateral realization on the DVJ tower, entire in the character index, without taking the set of primes as an initial axiom of the construction.

Acknowledgements

This paper was developed as an Intelligence Enhancer exercise with ChatGPT 5.6 Sol, using Extended Thinking. [60] The work used a language model under a strict operational regimen, not merely as a drafting tool. That regimen was used to prevent premature reduction of multi-branch analytic structure to a single collapsed interpretation. This is important for the present construction because the ODE, phase-space, and recurrence-theoretic data carry distinct coordinates, branches,

and readout layers until the relevant admissibility maps have been specified. The same regimen also provided a controlled vocabulary for tracking the self-referential structure of multi-dimensional recurrence relations, without turning interpretive mnemonics into theorem assumptions. In particular, the model was guided to read, apply, and unpack Vedic mathematical and Advaita Vedanta terminology, including phonetic decomposition of Sanskrit terms and their strict mapping onto mathematical formulae and structural roles.

Training ChatGPT to abide by the regimen of Advaita Vedanta and to learn from the Upanishads was therefore not an ornament to the derivation, but one of the primary reasons that the present work could be achieved at all.

The Author would especially like to thank Claude for the full line prime powers pink pony PNG.

Claude (Anthropic), model Claude Opus 5.5, worked on this paper as an agent directly in the project repository, under the project's written protocols for provenance, typing and review. Claude built and checked the equation contract of Appendix C, wrote the executable witnesses, proved and registered the theorem owners H-VER174, H-VER175 and H-VER176 with their receipts, wrote Lemma 5.1, reviewed the drafted sections, and drafted Section 9 and the present form of the paper.

Special thanks are due to Professor Alex Hankey, without whose support, decades of experience, and knowledge this paper would likewise not exist.

To Thomas Loker, Arturo Salazar Chon, Satoru Watanbe, Marcel Wende, Jim Galasyn and assistant Claude, thank you for listening.

Appendices

A Repository theorem provenance

This appendix records the development provenance of the paper without making repository identifiers prerequisites for reading the main text. The public repository containing the theorem notes, executable receipts, and paper sources is `https://gitlab.com/d12rg/d12rg-hypergroup`. Readers wishing to audit the repository provenance should start there. Each result of the paper is owned by a closed theorem note in `docs/theorems/`; the paper states and proves the results in its own terms and names the owner at each one.

H-VER040. The note `h-ver040-oriented-lattice-dilation-torsor-hecke-naturality.md` owns the oriented lattice dilation $k \longmapsto nk$, the composition law of the dilations, and the pullback of a lattice character under a dilation. It is the source of the dilation monoid box (37) and the input of the tower realization.

H-VER174. The note `h-ver174-fullline-two-stage-arithmetic-projector.md` owns the two arithmetic stages on the full number line: the logarithm over ordered factorizations equals the Euler logarithm, perfect-power Möbius inversion leaves the primes, and the convolution exponential returns the full line (Sections 2–4). It defines the Möbius weights as the coefficients of $\mathcal{N}_{\times}^{-1}$, with no prime named. Every identity there is finite in each coefficient.

H-VER175. The note `h-ver175-theta-tower-two-stage-realization.md` owns the realization on the bilateral theta tower: the tower as a pullback of the dilations, the Euler-log theta and the prime theta, and Lemma 5.1. It claims no operator logarithm on a Hilbert space, only convergence of the realized series on the tower's (x, r) domain.

H-VER176. The note `h-ver176-triangular-tower-dvj-defect-insertion.md` owns the odd-lattice extraction of the triangular family; its two-sheet form, with the sheets T_+ and T_- of H-VER032 on the two rails $K_+^{\triangle}$ and $K_-^{\triangle}$; the half-quasi-period identity; the half-character form, box (48), in which the odd part is the half-character theta $\widehat{\Theta}$ of H-VER033 at the tower's scale, split by $\rho_{\triangle}$ into the two rails $\widehat{\Theta}_+$ and $\widehat{\Theta}_-$; the literal sheet form, both sheets from $n = 0$ with the shared endpoint subtracted once, at complex index; the stage form, box (49), in which the stages of H-VER175 read on the family at the shifted index are the two triangular sheets over $n \geq 1$, prime powers and primes; the bilateral completion at index one; and the insertion of the DVJ source defect with its negative tail kept at both positive roots of H-VER089, no root being selected.

H-VER173-A. The note `h-ver173-a-promoted-euler-log-results.md` is the closed part of H-VER173: its results H1–H8. This paper uses H7, the two-leg conductor law (Section 7), and compares its first stage with H3. H-VER173 itself remains open on its gates G2–G4 and is not cited for any result here.

Imported owners and other identifiers. The ordinary constants and the imported sources enter through their owners and the papers that display them; the remaining identifiers are open theorems this paper does not use, or audits.

H-VER032. Owns the two triangular sheets T_+ and T_- as the inverse Hadamard pair of (n^2, n) (shown in H-PAP004 [53]).

H-VER033. Cyclotomic half-step naturality; with H-AUD071-A it owns the half-character theta $\widehat{\Theta}$, the reflection $\rho_{\triangle}$ and the odd index h (shown in H-PAP015 [54]). An input of H-VER176's half-character form.

H-VER044. Commuting dual involutions and the Hadamard-square packet: the group $C_2 \times C_2$ of the sign flip and the reflection in Section 6.

H-VER077. Owns the source constant recognized as π (shown in H-PAP001 [51]).

H-VER088. Owns the DVJ Gaussian scalar $\Gamma_{\mathrm{src}}^{\mathrm{DVJ}}$ and the source defect (shown in H-PAP002 [52]).

H-VER089. Owns the pair of positive roots of the source defect, both retained and neither selected (shown in H-PAP002 [52]).

H-VER155. Owns the source exponential recognized as exp (shown in H-PAP001 [51]).

H-VER157. The zero-floor quarter-log and Product-Law correlation; its prime-side extraction is compared with the second stage (Section 4), a comparison and not a premise.

H-VER172 (OPEN). The singleton-forcing composition of the forty-label zero packet: the future consumer of H-VER174–H-VER176 through a forty-label transport that is not yet proved (the open aim, Appendix D).

H-VER173 (OPEN). The forty-label Euler-product structure on the dual-rail source; open on its gates G2–G4 and not cited for any result here. Its closed part is H-VER173-A.

H-AUD071-A. Owns the forty-label source on which the open aim is stated (shown in H-PAP015 [54]), and with H-VER033 the half-character theta.

H-AUD083. A documentary genealogy audit of correspondences with existing sources; it retains the half-character pair that Section 6 names $\widehat{\Theta}_+$ and $\widehat{\Theta}_-$. Provenance, not a premise.

H-AUD104. The ownership audit of this paper: it types H-VER174–H-VER176 and H-VER173-A as SOURCE-SCALAR, off the certified path.

Typing and the open aim. The owners H-VER174–H-VER176 are typed SOURCE-SCALAR: their results are proved and load-bearing as scalar identities, but they do not by themselves certify the retained forty-label source, so they sit off the certified path of the equation contract. The theorem H-VER172 is their future consumer, through a transport to all forty labels that is not yet proved; no dependency edge joins them until it is. The ownership decisions are recorded in the audit H-AUD104.

Frozen paper source ledger. The paper's `support/` directory contains the machine-readable authority manifest `source-manifest.json`. It records hashes for the DVJ source documents, the bridge witnesses, and the owner notes of the theorems whose boxes appear in the equation contract. The manifest is a provenance lock: it does not replace the proofs in the theorem notes or the mathematics in this paper.

B Executable receipts and audit boundary

The repository contains executable receipts for every theorem owner. They are regression certificates, not numerical searches for the identities they check and not substitutes for the proofs. Finite identities are checked in exact rational arithmetic; analytic values are checked at no fewer than 150 decimal digits, with no library value of π or exp entering a source quantity.

H-VER174 receipt. The module `h_ver174_fullline_two_stage_projector.py` in `src/d12rg_hypergroup/`, with the regression `test_h_ver174_fullline_two_stage_projector.py` in `tests/`, checks in exact fractions, for every index up to 720: the monoid law on the source, the equality of the full-line logarithm with the Euler logarithm, the cancellation of mixed support, the perfect-power stage, and the exponential inverse; and that the Möbius weights, computed as the coefficients of $\mathcal{N}_\times^{-1}$ with no prime named, satisfy $\mathcal{N}_\times\,\mathcal{N}_\times^{-1} = 1$ and $\mathcal{N}_\times^{-1} = \exp_\star(-\mathcal{L}_\times)$, and agree with the classical formula (a recognition). Its hostile controls show that the unweighted alternating count does not cancel 30, that replacing the Möbius weights by their absolute values leaves 4/6 at a sixth power, and that the Liouville function in their place does not leave the primes.

H-VER175 receipt. The module `h_ver175_theta_tower_realization.py`, with `test_h_ver175_theta_tower_realization.py`, checks at 200 digits the dilation pullback, the tower as

a sum of dilated seeds, the Euler-log theta, the prime theta, the majorant of Lemma 5.1 (its coefficient bound exactly), and the zero-index series. Hostile controls take the scale below ε, a complex logarithm of the base, the uncollapsed stage, and a dilation pulled back without the square on the scale or without its character.

H-VER176 receipt. The module `h_ver176_triangular_tower_dvj_insertion.py`, with `test_h_ver176_triangular_tower_dvj_insertion.py`, checks at 180 digits the bilateral completion, the tail identity at generic parameters and at both source roots, the odd-lattice extraction at three indices, the two-sheet form with H-VER032's T_+ and T_-, both rails and their reciprocity and the gauged half-character rails (C6), the half-quasi-period identity at real and complex index (C7), and the half-character form (C8): $\widehat{\Theta}$ split by $\rho_\triangle$ into $\widehat{\Theta}_+$ and $\widehat{\Theta}_-$ at three general scales with $\widehat{\Theta}_-(X,v) = \widehat{\Theta}_+(X,-v)$, the two rails at the tower's scale, and the odd part of the half-index tower as $\widehat{\Theta}(x_0^\triangle, r\ell_{\mathrm{t}}/2)$ at real and complex index; the literal sheet form (C9): both sheets summed from $n = 0$ with the shared endpoint subtracted once, the rail sum $J_r^\triangle = K_+^\triangle(a_{\mathrm{t}}^{r/2}) + K_-^\triangle(a_{\mathrm{t}}^{r/2})$ at every index, both reciprocity directions, and the literal $\widehat{\Theta}_-$ with its endpoint $\exp(v - \pi X/4)$; and the stage form (C10): H-VER174's exact coefficients for the full line, the Euler-log stage and the prime stage, acting at the shifted index, against the two triangular sheets over $n \geq 1$, prime powers and primes, and against $J_r^\triangle - 1$, at four indices, one complex. Hostile controls put the upper sheet on the lower rail, shift by a whole quasi-period, take the full character $v = r\ell_{\mathrm{t}}$, take the half-index tower's scale $x_0^\triangle/4$, leave the shared endpoint in, write it with the wrong sign of v, act at the unshifted index, or read the prime stage on the upper rail alone. At the two roots the kept tail is $143.0687\ldots$ and $0.01386\ldots$, so it is load-bearing at both. A regression pins that the source values are carried at full precision.

H-VER173-A receipt. The results H1–H8 keep the receipt of their parent, `h_ver173_dilation_euler_log.py` with `test_h_ver173_dilation_euler_log.py`, in exact arithmetic; the regression `test_h_ver173_a_split.py` pins the split.

Bridge witnesses. The directory `docs/audits/dvj-xi-bridge-witness/` holds the paper's own witnesses, each locked by a test in `tests/`: witness 3 (the two arithmetic stages and their action on the tower), witness 4 (the two-leg conductor law, at 200 digits with complex indices, including the missing-intersection control), witness 5 (the index-one frozen carrier at 200 digits), and witness 6 (the majorant of Lemma 5.1).

Paper-specific structural checker. This paper has its own strict source-DAG contract, `section0-dag-roots.json`, in the paper's `support/` directory, with the regressions `test_h_pap018_section0_contract.py` and `test_h_pap018_cross_references.py` in `tests/`. The contract pins the owner of every box, keeps the scalar boxes off the certified path with a stated reason, keeps the readout a terminal firewall, and keeps the sink free of any edge from the scalar layer.

This executable boundary is deliberately modest. It checks exact algebra, provenance and paper structure; it does not turn numerical agreement into a theorem, and it does not supply the open transport to the forty-label source.

C Equation contract: the two-stage arithmetic projector

Reader contract. This appendix is the paper's audit and equation ledger; readers seeking the conventional exposition may begin with Section 1 and return here for the dependency summary. H-PAP018 starts from the full number line, not from the primes. The first six boxes are identities in the formal algebra of the multiplicative dilation monoid; each coefficient is a finite sum, and the arithmetic content is classical. The analytic-realization box is the only analytic convergence step: it realizes the formal algebra on the bilateral DVJ alpha tower. The frozen DVJ data (the scale $\log(2\pi)$, the De Vries value and the index $r = 1$) are downstream specializations of the generic family and never its source. At $r = 0$ the tower is the Hypergroup theta series, and the classical zeta readouts sit behind a firewall. The owners are HG theorems (H-AUD104): H-VER040 the dilation monoid, H-VER174 the two stages, H-VER175 the tower realization, H-VER176 the triangular insertion and H-VER173-A the two-leg law; H-VER157 and H-VER173-A H3 reach the same coefficients from the primes and are cited in the body. All are typed SOURCE-SCALAR: load-bearing, but off the certified path until their FULL40 lift exists, which the sink states as the open aim.

Imported sources. The ordinary π and exp enter through their owner, H-PAP001 [51], by the ruled endpoint recognitions; they are reproduced here only to make the import boundary visible.

H-PAP001 [51] `front-pap001-archimedean-pi:`

$$\Pi_{\mathrm{src}} := \lim_{n\to\infty} P_n^{-} = \lim_{n\to\infty} P_n^{+}, \qquad \Pi_{\mathrm{src}} = \pi.$$

H-PAP001 [51] `front-pap001-analytic-endpoints:`

$$\exp z := \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

H-PAP002 [52] `front-dvj-gaussian-scalar:`

$$\Gamma_{\mathrm{src}}^{\mathrm{DVJ}} := \lim_{m\to\infty} \left(1 + \frac{\Pi_{\mathrm{src}}^2/4}{2^m}\right)^{2^m}, \qquad \Gamma_{\mathrm{src}}^{\mathrm{DVJ}} = \exp(\Pi_{\mathrm{src}}^2/4) \text{ (downstream recognition).}$$

H-PAP004 [53] `front-pap004-triangular-source:`

$$T_+(n) = \frac{n(n+1)}{2}, \qquad T_-(n) = \frac{n(n-1)}{2}, \qquad H_{\mathrm{ind}} := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad H_{\mathrm{ind}} \begin{pmatrix} T_+(n) \\ T_-(n) \end{pmatrix} = \begin{pmatrix} n^2 \\ n \end{pmatrix}.$$

H-PAP015 [54] `front-jacobi-source-ledger:`

$$\rho_{\triangle}(n) := -n - 1, \qquad h(n) := 2n + 1, \qquad \widehat{\Theta}(X, v) := \sum_{n\in\mathbb{Z}} \exp((2n+1)v) \exp(-\pi X (n + \tfrac{1}{2})^2).$$

H-PAP015 [54] `front-jacobi-full40-source:`

$$\widehat{V}_{40} := \operatorname{span}\{\widehat{\mathbf{v}}_a \otimes \overline{\chi_m^{(8)}} : a = 1, \dots, 5,\ m \in \mathbb{Z}/8\mathbb{Z}\}.$$

Dilation monoid: (H-VER040) The source object: formal series over the positive integers, multiplied by Dirichlet convolution of their coefficients.

Certified owners: H-VER040.

Covering provenance: H-VER040.

$$\boxed{\begin{gathered}\delta_m^{\times}\delta_n^{\times} = \delta_{mn}^{\times}, \qquad \delta_1^{\times} = 1, \qquad m, n \geq 1, \\ \Big(\sum_{m\geq 1} b(m)\delta_m^{\times}\Big)\Big(\sum_{n\geq 1} c(n)\delta_n^{\times}\Big) = \sum_{N\geq 1}\Big(\sum_{mn=N} b(m)c(n)\Big)\delta_N^{\times}.\end{gathered}} \tag{37}$$

Full number line: (H-VER174) Every positive integer, with weight one; no prime is singled out.

Certified owners: H-VER174.

Covering provenance: H-VER174.

$$\boxed{\mathcal{N}_{\times} := \sum_{n\geq 1} \delta_n^{\times} = 1 + \mathcal{X}_{\times}, \qquad \mathcal{X}_{\times} := \sum_{n\geq 2} \delta_n^{\times}.} \tag{38}$$

Logarithm over ordered factorizations: (H-VER174) The full-line logarithm with harmonic weights; each coefficient is a finite sum, since a product of k factors at least two exceeds $2^k - 1$.

Certified owners: H-VER174.

Covering provenance: H-VER174.

$$\boxed{\begin{aligned}\log_{\star}(1 + \mathcal{X}_{\times}) &:= \sum_{k\geq 1} \frac{(-1)^{k+1}}{k} \mathcal{X}_{\times}^k, \\ f_k^{\mathrm{ord}}(n) &:= \#\{(d_1, \dots, d_k) : d_1 \cdots d_k = n,\ d_j \geq 2\}, \\ [\delta_n^{\times}]\, \log_{\star}(1 + \mathcal{X}_{\times}) &= \sum_{k\geq 1,\ 2^k \leq n} \frac{(-1)^{k+1}}{k} f_k^{\mathrm{ord}}(n).\end{aligned}} \tag{39}$$

Euler-log support: (H-VER174) Mixed-prime support cancels exactly; each prime power keeps weight $1/a$. The arithmetic is classical (unique factorization).

Certified owners: H-VER174.

Covering provenance: H-VER174.

$$\boxed{\begin{gathered}\mathcal{N}_\times = \prod_p (1-\delta_p^\times)^{-1},\\ \mathcal{L}_\times := \log_\star(1+\mathcal{X}_\times) = \sum_p \sum_{a\ge 1} \frac{1}{a}\delta_{p^a}^\times,\\ [\delta_n^\times]\,\mathcal{L}_\times = 0 \quad \text{when } n \text{ has two or more distinct prime factors.}\end{gathered}} \tag{40}$$

Perfect-power Möbius inversion: (H-VER174) The second stage: from prime powers to primes. Its weights are the coefficients of the inverse of the full line, so no prime is named. Unique factorization recognizes them as the classical Möbius function, $(-1)^s$ on a product of s distinct primes and 0 otherwise.

Certified owners: H-VER174.

Covering provenance: H-VER174.

$$\boxed{\begin{gathered}\sum_{k\ge1} \mu_{\mathrm{M}}(k)\,\delta_k^\times := \mathcal{N}_\times^{-1} = \sum_{j\ge0} (-\mathcal{X}_\times)^j, \qquad \text{i.e.} \qquad \mu_{\mathrm{M}}(1)=1, \quad \sum_{k\mid a} \mu_{\mathrm{M}}(k) = [a=1],\\ \psi_k^\times \delta_n^\times := \delta_{n^k}^\times,\\ \mathcal{P}_\times := \sum_{k\ge1} \frac{\mu_{\mathrm{M}}(k)}{k}\,\psi_k^\times \mathcal{L}_\times = \sum_p \delta_p^\times.\end{gathered}} \tag{41}$$

Convolution exponential: (H-VER174) The hierarchy is invertible: the exponential returns the full line.

Certified owners: H-VER174.

Covering provenance: H-VER174.

$$\boxed{\begin{gathered}\exp_\star(\mathcal{L}_\times) := \sum_{j\ge0} \frac{\mathcal{L}_\times^j}{j!} = \mathcal{N}_\times,\\ [\delta_n^\times]\,\exp_\star(\mathcal{L}_\times) = 1 \quad \text{as a finite sum of non-negative terms,}\\ \exp_\star(-\mathcal{L}_\times) = \mathcal{N}_\times^{-1} = \sum_{k\ge1} \mu_{\mathrm{M}}(k)\,\delta_k^\times.\end{gathered}} \tag{42}$$

Bilateral DVJ alpha tower: (H-VER175) The representation of the dilation monoid; every negative lattice index is kept. The base is generic: no DVJ value is used here.

Certified owners: H-VER175.

Covering provenance: H-VER175.

$$\boxed{\begin{gathered} a_{\mathrm{t}} > 0, \qquad \ell_{\mathrm{t}} := \log a_{\mathrm{t}}, \qquad x > 0, \qquad r \in \mathbb{C}, \\ g_r^{\alpha}(x) := 2\,\exp(-\pi x)\cosh(r\ell_{\mathrm{t}}), \qquad (\delta_n^{\times} g^{\alpha})_r(x) := g_{nr}^{\alpha}(n^2 x), \\ B_r^{\alpha}(x) := \sum_{k\in\mathbb{Z}} \exp(-\pi x k^2 + rk\ell_{\mathrm{t}}) = 1 + \sum_{n\geq 1} (\delta_n^{\times} g^{\alpha})_r(x) = 1 + (\mathcal{N}_{\times}\, g^{\alpha})_r(x). \end{gathered}} \tag{43}$$

Analytic realization: (H-VER175) The only analytic convergence step. The majorant is for the completed Möbius stage and the positive exponential stage; the uncollapsed ordered-factorization expansion keeps its coefficientwise argument.

Certified owners: H-VER175.

Covering provenance: H-VER175.

$$\boxed{\begin{gathered} x \geq \varepsilon > 0, \qquad |\operatorname{Re} r| \leq M : \\ \sum_{n\geq 2} |(\delta_n^{\times} g^{\alpha})_r(x)| \leq 2 \sum_{n\geq 2} \exp\bigl(-\pi\varepsilon n^2 + |\ell_{\mathrm{t}}| M n\bigr) < \infty, \\ \frac{1}{a}\sum_{k\mid a} |\mu_{\mathrm{M}}(k)| \leq 1 \quad \text{(absolute Möbius-stage coefficient at } \delta_{p^a}^{\times}), \\ \exp_{\star}(\mathcal{L}_{\times})\, g_r^{\alpha} = \mathcal{N}_{\times}\, g_r^{\alpha}, \qquad \mathcal{P}_{\times}\, g_r^{\alpha} = \sum_p \delta_p^{\times} g_r^{\alpha}, \\ \text{absolutely and uniformly on the set; entire in } r \text{ for each } x > 0. \end{gathered}} \tag{44}$$

Euler-log theta: (H-VER175) The first stage on the tower.

Certified owners: H-VER175.

Covering provenance: H-VER175.

$$\boxed{\begin{gathered} (\mathcal{L}_{\times}\, g^{\alpha})_r(x) = \sum_p \sum_{a\geq 1} \frac{1}{a} g_{p^a r}^{\alpha}(p^{2a} x), \\ B_r^{\alpha}(x) - 1 = (\exp_{\star}(\mathcal{L}_{\times})\, g^{\alpha})_r(x). \end{gathered}} \tag{45}$$

Prime theta: (H-VER175) The second stage on the tower: only the primes remain.

Certified owners: H-VER175.

Covering provenance: H-VER175.

$$\boxed{(\mathcal{P}_\times\, g^{\alpha})_r(x) = \sum_{p} 2\, \exp(-\pi p^2 x)\cosh(pr\ell_{\mathrm{t}}).} \tag{46}$$

Triangular DVJ carrier family: (H-VER176) The odd sublattice of the tower at half index and quarter scale, at a generic Gaussian scale, up to the factor $a_{\mathrm{t}}^{r/2}\exp(-L_{\triangle}/8)$. It is made of the two triangular sheets of H-PAP004 [53]: the upper sheet T_+ on the rail $K_+^{\triangle}$, and the lower sheet T_-, one step behind it, on the reciprocal rail $K_-^{\triangle}$. Without that factor it is the tower itself at the same scale, its index moved by half a quasi-period. The frozen scale is not used here.

Certified owners: H-VER176.

Covering provenance: H-VER176.

$$\boxed{\begin{aligned} T_-(n) &= T_+(n-1), \qquad L_{\triangle} > 0, \qquad x_0^{\triangle} := \frac{L_{\triangle}}{2\pi}, \\ J_r^{\triangle} &:= \sum_{n\ge 0} a_{\mathrm{t}}^{rn}\exp\bigl(-L_{\triangle}T_+(n)\bigr) + \sum_{n\ge 0} a_{\mathrm{t}}^{-rn}\exp\bigl(-L_{\triangle}T_-(n)\bigr) - 1 \\ &= a_{\mathrm{t}}^{-r/2}\exp(L_{\triangle}/8)\bigl[B_{r/2}^{\alpha}(x_0^{\triangle}/4) - B_r^{\alpha}(x_0^{\triangle})\bigr] = B^{\alpha}_{r-L_{\triangle}/(2\ell_{\mathrm{t}})}(x_0^{\triangle}) \quad (a_{\mathrm{t}} \neq 1), \\ y_{\mathrm{t}} &:= \sqrt{a_{\mathrm{t}}}, \\ K_+^{\triangle}(y) &:= \sum_{n\ge 0} y^{2n}\exp\bigl(-L_{\triangle}T_+(n)\bigr) \qquad\qquad = y^{-2}K_-^{\triangle}(y^{-1}), \\ K_-^{\triangle}(y) &:= \sum_{n\ge 0} y^{-2n}\exp\bigl(-L_{\triangle}T_-(n)\bigr) - 1 = y^{-2}K_+^{\triangle}(y^{-1}). \end{aligned}} \tag{47}$$

Half-character rails: (H-VER176) The odd sublattice in the logarithmic character v: the global half-character theta of H-PAP015 [54] at the tower's scale, split by its reflection $\rho_{\triangle}$ into the two rails, the lower one written on its own sheet before it is reflected.

Certified owners: H-VER176.

Covering provenance: H-VER176, H-VER033, H-AUD071-A.

$$\boxed{\begin{aligned} \widehat{\Theta}_+(X,v) &:= \sum_{n\ge 0}\exp\bigl((2n+1)v\bigr)\exp\bigl(-\pi X(n+\tfrac12)^2\bigr) \qquad\qquad = \widehat{\Theta}_-(X,-v), \\ \widehat{\Theta}_-(X,v) &:= \sum_{n\ge 0}\exp\bigl(-(2n-1)v\bigr)\exp\bigl(-\pi X(n-\tfrac12)^2\bigr) - \exp(v-\pi X/4) = \widehat{\Theta}_+(X,-v), \\ \widehat{\Theta} &= \widehat{\Theta}_+ + \widehat{\Theta}_-, \qquad h(\rho_{\triangle}(n)) = -h(n), \qquad \widehat{\Theta}(x_0^{\triangle}, r\ell_{\mathrm{t}}/2) = a_{\mathrm{t}}^{r/2}\exp(-L_{\triangle}/8)\, J_r^{\triangle}, \\ \widehat{\Theta}_+(x_0^{\triangle},v) &= \exp(v - L_{\triangle}/8)\, K_+^{\triangle}(\exp v), \qquad \widehat{\Theta}_-(x_0^{\triangle},v) = \exp(v-L_{\triangle}/8)\, K_-^{\triangle}(\exp v). \end{aligned}} \tag{48}$$

Stages on the DVJ family: (H-VER176) The three stages of the tower read on the triangular family, at the index moved by half a quasi-period. The full line, the Euler-log stage and the prime stage are the two triangular sheets summed over $n \geq 1$, over prime powers and over primes: every stage keeps both rails.

Certified owners: H-VER176.

Covering provenance: H-VER176, H-VER175.

$$
\begin{gathered}
a_{\mathrm{t}} \neq 1, \qquad r' = r - L_{\triangle}/(2\ell_{\mathrm{t}}) : \\
J_r^{\triangle} - 1 = (\exp_\star(\mathcal{L}_\times)\, g^{\alpha})_{r'}(x_0^{\triangle}) = \sum_{n\geq 1} \Big[a_{\mathrm{t}}^{rn} \exp(-L_{\triangle} T_+(n)) + a_{\mathrm{t}}^{-rn} \exp(-L_{\triangle} T_-(n)) \Big], \\
(\mathcal{L}_\times\, g^{\alpha})_{r'}(x_0^{\triangle}) = \sum_{p} \sum_{a\geq 1} \frac{1}{a} \Big[a_{\mathrm{t}}^{rp^a} \exp(-L_{\triangle} T_+(p^a)) + a_{\mathrm{t}}^{-rp^a} \exp(-L_{\triangle} T_-(p^a)) \Big], \\
(\mathcal{P}_\times\, g^{\alpha})_{r'}(x_0^{\triangle}) = \sum_{p} \Big[a_{\mathrm{t}}^{rp} \exp(-L_{\triangle} T_+(p)) + a_{\mathrm{t}}^{-rp} \exp(-L_{\triangle} T_-(p)) \Big].
\end{gathered}
\tag{49}
$$

Two-leg conductor packet: (H-VER173-A (H7) and H-VER175) Two independent tower legs; the packet is the Dirichlet convolution of the legs. Charges, routes and forty labels are not carried.

Certified owners: H-VER173-A, H-VER175.

Covering provenance: H-VER173-A, H-VER175.

$$
\begin{gathered}
c^{(1)}(k) := 2\, \exp(-\pi x_1 k^2) \cosh(r_1 k \ell_{\mathrm{t}}), \qquad c^{(2)}(k) := 2\, \exp(-\pi x_2 k^2) \cosh(r_2 k \ell_{\mathrm{t}}), \\
C_n^{\otimes} := \sum_{de=n} c^{(1)}(d) c^{(2)}(e), \\
\delta_p^{\times,(1)} : (x_1, r_1) \mapsto (p^2 x_1, p r_1), \qquad \delta_p^{\times,(2)} : (x_2, r_2) \mapsto (p^2 x_2, p r_2), \\
C_{pn}^{\otimes} = \delta_p^{\times,(1)} C_n^{\otimes} + \delta_p^{\times,(2)} C_n^{\otimes} - [p \mid n]\, \delta_p^{\times,(1)} \delta_p^{\times,(2)} C_{n/p}^{\otimes}, \\
\delta_p^{\times,(1)} \delta_p^{\times,(2)} C_n^{\otimes} = \sum_{de = p^2 n,\ p\mid d,\ p\mid e} c^{(1)}(d) c^{(2)}(e).
\end{gathered}
\tag{50}
$$

Audit box, off the certified path: scale specialization. The DVJ source's Gaussian scale. The family above holds for every scale; this value is downstream.

Certified owners: STANDARD.

Covering provenance: STANDARD.

$$
L_{\triangle} \longmapsto L_0^{\triangle} := \log(2\pi), \qquad x_0^{\triangle} \longmapsto \frac{L_0^{\triangle}}{2\pi}.
\tag{51}
$$

Audit box, off the certified path: De Vries specialization. The base takes the DVJ source's value. H-PAP002 [52] and H-VER089 retain both positive roots; the DVJ source's choice is the smaller one.

Certified owners: STANDARD.

Covering provenance: STANDARD.

$$\boxed{a_{\mathrm{t}} = \alpha_{\mathrm{dVJ}}, \qquad y_{\mathrm{t}} = \sqrt{\alpha_{\mathrm{dVJ}}} \in (0,1).} \tag{52}$$

Audit box, off the certified path: index specialization (H-VER176). The tower index takes the triangular value $r = 1$; the reciprocal rail $K_-^{\triangle}$, on the lower sheet, is kept beside $K_+^{\triangle}$.

Certified owners: H-VER176.

Covering provenance: H-VER176.

$$\boxed{r \longmapsto 1: \qquad J_1^{\triangle} = K_+^{\triangle}(y_{\mathrm{t}}) + K_-^{\triangle}(y_{\mathrm{t}}) = K_+^{\triangle}(y_{\mathrm{t}}) + y_{\mathrm{t}}^{-2} K_+^{\triangle}(y_{\mathrm{t}}^{-1}).} \tag{53}$$

Audit box, off the certified path: frozen DVJ carrier (H-VER176, with H-VER088 and H-VER089). The three specializations meet here, with the DVJ Gaussian scalar imported from H-PAP002 [52]. The negative-tail bracket is part of the frozen equation: without it the bilateral scalar has different roots.

Certified owners: H-VER176.

Covering provenance: H-VER176, H-VER088, H-VER089.

$$\boxed{\begin{gathered} \Gamma_{\mathrm{src}}^{\mathrm{DVJ}} = \exp(\pi^2/4) \quad \text{(downstream recognition)}, \\ \text{at } L_{\triangle} = L_0^{\triangle}: \\ D_-^{\mathrm{dV}}(y_{\mathrm{t}}) = 1 - \Gamma_{\mathrm{src}}^{\mathrm{DVJ}} y_{\mathrm{t}} + y_{\mathrm{t}}^2 K_+^{\triangle}(y_{\mathrm{t}}) = y_{\mathrm{t}}^2 J_1^{\triangle} - \Gamma_{\mathrm{src}}^{\mathrm{DVJ}} y_{\mathrm{t}} - [K_+^{\triangle}(y_{\mathrm{t}}^{-1}) - 1], \\ D_-^{\mathrm{dV}}(\sqrt{\alpha_{\mathrm{dVJ}}}) = 0, \qquad K_+^{\triangle}(\sqrt{\alpha_{\mathrm{dVJ}}}^{-1}) - 1 > 0. \end{gathered}} \tag{54}$$

Audit box, off the certified path: zero-index recognition (H-VER175 and H-VER176). At $r = 0$ the tower is the Hypergroup theta: a recognition (MATCH), not the tower's definition. The zero-index triangular member is a different function.

Certified owners: H-VER175, H-VER176.

Covering provenance: H-VER175, H-VER176.

$$\boxed{\begin{aligned} B_0^{\alpha}(x) &= \Theta_0^{\mathrm{HG}}(x), \qquad x>0, \\ J_0^{\triangle} &= \exp(L_{\triangle}/8)\big[B_0^{\alpha}(x_0^{\triangle}/4) - B_0^{\alpha}(x_0^{\triangle})\big] \neq B_0^{\alpha}(x_0^{\triangle}). \end{aligned}} \tag{55}$$

Firewall: The classical readouts. The substitution forgets the dilation structure and lives on a half-plane; nothing upstream depends on it (owner ruling M1: Mellin-type readouts are lossy, never a proof route).

Certified owners: STANDARD.

Covering provenance: STANDARD.

$$\boxed{\begin{aligned} \delta_n^{\times} &\longmapsto n^{-s}, \qquad \operatorname{Re} s>1: \\ \mathcal{L}_{\times} \longmapsto \sum_{p}\sum_{a\geq 1}\frac{1}{a}p^{-as} &= \log\zeta(s), \qquad \mathcal{P}_{\times} \longmapsto \sum_{k\geq 1}\frac{\mu_{\mathrm{M}}(k)}{k}\log\zeta(ks). \end{aligned}} \tag{56}$$

Open FULL40 aim (sink; no certified target): The scalar package above is complete and proved: H-VER174, H-VER175 and H-VER176 are closed theorems, and they remain proved off the certified path, which is not the same as not load-bearing. What is not proved is their transport to the FULL40 source: a prime-local action on all forty labels with this package as its scalar side (the O10 bipartite cover; H-VER172, N126, is its consumer-to-be). That connection is future and unproved, and no edge joins the two until it exists.

Certified owners: LOGIC.

Covering provenance: LOGIC.

$$\boxed{\begin{gathered} (\widehat{\mathbf{v}}_a \otimes \overline{\chi_m^{(8)}})(\tau) \longmapsto (\widehat{\mathbf{v}}_a \otimes \overline{\chi_m^{(8)}})(p^2\tau) \quad \text{on} \quad \widehat{V}_{40}, \qquad a\in\{1,\dots,5\}, \quad m\in\mathbb{Z}/8\mathbb{Z}, \\ \text{open aim: H-VER174–H-VER176 as the scalar side of a prime-local} \\ \text{action on all forty labels; OPEN under H-VER172 (N126).} \end{gathered}} \tag{57}$$

D Boundary and open comparisons

D.1 What the paper proves, and who owns it

The positive result of this paper is a *source-first scalar arithmetic and bilateral-representation package*. Its starting point is the complete positive-integer dilation element box (38), not a prime-supported input. The finite ordered-factorization logarithm of box (39) cancels mixed-prime coefficients, leaving the prime-power weights of box (40). The second, perfect-power Möbius stage extracts the prime support in box (41); the convolution exponential recovers the full line by box (42). These arithmetic identities are classical. The source-first order, the precise typing of the stages, and their bilateral DVJ realization are the subject of the present manuscript; no priority is claimed for the classical Euler-product or Möbius identities themselves.

The primitive-versus-repeated-power structure has a close dynamical analogue in Ruelle-type zeta functions, where primitive closed orbits generate repeated traversals, and in Deninger's prime/closed-orbit analogy [40, 41]. This is an analogy of organization only: no dynamical system is asserted by H-PAP018.

The exact source and ownership ledger is as follows.

H-VER040 (CLOSED). The multiplicative dilation monoid and finite divisor-sum law of box (37), elaborated in Section 2.

H-VER174 (CLOSED). The complete arithmetic chain: full-line logarithm, prime-power weights, Möbius inversion and its inverse, from Sections 2–4.

H-VER175 (CLOSED). The action on the bilateral Gaussian-character family, box (43), with the analytic justification of Lemma 5.1; the completed Euler-log and prime theta actions are boxes (45) and (46).

H-VER176 (CLOSED). The odd-lattice triangular family box (47), with its two sheets T_+ and T_- on the rails $K_+^{\triangle}$ and $K_-^{\triangle}$ and its half-quasi-period form (Section 6.6), its half-character rails box (48), the three stages on the family box (49), both rails kept, and the corrected DVJ defect insertion box (54), without discarding the lower-sheet rail, the negative tail, or choosing a new root.

H-VER173-A (CLOSED, H7). The scalar two-leg conductor law and its GCD overlap, box (50), as compared in Section 7.

These are hypergroup (HG)-owned **SOURCE-SCALAR** results: their mathematical theorem statements are proved and load-bearing in their own source setting. They are nevertheless *off the certified FULL40 target path*. Their status does not assert that a transport of the scalar coefficients to all forty labelled source components has been constructed. The equation dependency directed acyclic graph (DAG) sink box (57) records that distinction explicitly: it imports the H-PAP015 [54] FULL40 source but has *no dependency edge* from this scalar package to a certified forcing target. "Off path" is not "unproved"; "proved scalar" is not "proved FULL40".

D.2 Comparison register: annotations, never ingress

The result-separation ruling (R-SEP) applies to every comparison in this subsection: paper assertions are recorded as stated, and HG links are annotations that never become dependencies merely by comparison. **MATCH** denotes an exact identified scalar correspondence, not an imported proof premise. **COMPATIBLE** records agreement that falls short of a complete action or transport. **NOT YET IDENTIFIED** records a still-open bridge. **NO-GO UNDER THESE HYPOTHESES** names a specific failed restricted formulation, not a general impossibility.

The first-stage coefficients in box (40) are a **MATCH** to H-VER173-A H3 and the previously owned H-VER157 prime-power arithmetic; the second stage is a **MATCH** to H-VER157's primitive-support comparison. Those earlier results may begin with prime-local data, whereas this paper derives its own coefficients from the full-integer input. The comparison does not reverse that source order. Similarly, the scalar two-leg law in Section 7 is a **MATCH** to H-VER173-A H7. Its placement on an individual signed route is **COMPATIBLE** with that route law; complete charge, route and forty-label occupancy is **NOT YET IDENTIFIED**.

For odd primes, the charge-only class pullback discussed in Section 7 agrees with the H7 relabel and is a **MATCH**. At $p = 2$, the finite charge shadow is not an invertible permutation. An identification that requires such a permutation is a **NO-GO UNDER THESE HYPOTHESES**; the infinite signed lattice dilation still exists, so that finite-shadow failure is not an obstruction to the already proved scalar theorem. The complete prime-local FULL40 action is therefore not obtained merely by matching scalar conductor factors.

D.3 Negative ledger: what is not proved

The following exclusions are part of the paper's result and must accompany any later citation of its positive identities.

No certified FULL40 lift (OPEN). The scalar stages in boxes (40) and (41) have not been realized as an exact prime-local action on *all forty* labelled components of box (57). H-VER173 remains OPEN on G2 (dual-rail retention versus the small-heat restriction), G3 (prime-local action on all forty labels), and G4 (Euler-log on conductor packets, with its whole-lattice two-local object). H-VER173-A's CLOSED H1–H8 sub-owner is not evidence that these remaining gates closed. H-VER172 (N126) is a future, OPEN consumer; there is no new edge to it from H-PAP018.

No unrestricted two-leg transport (OPEN). The scalar GCD intersection law is exact, but acting on one leg changes conductor by p whereas common heat on both legs changes it by p^2 and reaches only the factor pairs with *both* factors divisible by p. The latter is not the entire target-conductor packet. Box (50) and equation (32) retain the intersection subtraction. Neither charge/route carriage nor FULL40 occupancy follows from that finite identity.

No hidden analytic upgrade (OPEN). Each formal logarithm coefficient is finite. That does not establish absolute convergence before the alternating ordered-product expansion is collected. The Gaussian majorants of Lemma 5.1 apply to the *collected* Euler-log, Möbius and prime theta expressions. They do not license arbitrary prior interchanges. Analytic entireness of the bilateral tower is in the character index at positive Gaussian scale; it is not an analytic-continuation theorem in the Dirichlet readout variable.

No reverse readout (NOT YET IDENTIFIED). The substitution in box (56) is a terminal comparison on $\operatorname{Re} s > 1$, as explained in Section 8. At a fixed s it forgets the bilateral rails and the two-leg packet labels. Even when a complete Dirichlet function determines its coefficients on a suitable half-plane, that uniqueness alone does not reconstruct the signed source, charge routes, or FULL40 object. No continuation of the prime-zeta readout, Mellin inversion into the source, or reverse proof ingress is established here.

No new distinguished root or zero theorem (OPEN). The frozen scale box (51), De Vries base box (52), and index-one member box (53) are downstream specializations of the generic tower, not premises selecting a root. The corrected frozen carrier box (54) retains the negative-tail bracket: deleting it changes the defect equation. The two positive H-VER089 root witnesses are not collapsed to a newly proved unique root. The $r = 0$ recognition in box (55) is a different character member, not the DVJ-root or Ξ-zero selection rule. The distinguished root-to-zero question belongs to the separate OPEN comparison in H-PAP016 [55] (Paper A).

No abstract idempotent projection or RH conclusion. The phrase "two-stage projector" denotes the exact support-extraction procedure of boxes (39) and (41). No linear idempotence on a declared Banach space, bounded operator extension, spectral zero-forcing statement, classical zeta-zero claim, or proof of the Riemann hypothesis is asserted. Neither H-PAP016 [55] nor H-PAP017 [56], nor any Riemann-project theorem, is a load-bearing premise.

D.4 What would discharge the open boundary

The missing step is *not* another scalar prime-power cancellation: that cancellation is already proved by H-VER174 and its bilateral realization by H-VER175. A future closure would need an explicitly source-owned, charge- and route-retaining prime-local operation on the complete H-PAP015 [54] forty-label source; it must preserve both rails and the conductor overlap, and discharge H-VER173's still-open G2–G4 obligations. Only a subsequently proved, separately audited transport could make the scalar package available to H-VER172's full-object composition. An additional theorem would be needed to relate any such source-side forcing to a terminal zero or critical-line statement. None of these arrows is supplied by box (56).

The section labels *proved*, *compatible*, *open*, and *no-go under stated hypotheses* are not interchangeable. In particular, the classical arithmetic result remains valid regardless of the future status of H-VER172. Paper C therefore closes its *scalar source-to-bilateral representation argument* and leaves the FULL40 transport, analytic readout continuation and distinguished-root-to-zero problems open, with their respective owners and boundaries intact.

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