Weyl Characters, Verlinde Fusion, and Generalized Chebyshev Hypergroups: An Exact Rank-Ten $A_3^{(1)}$ Level-Two Isomorphism

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Abstract

We prove an exact isomorphism between two independently constructed rank-ten multiplication systems: the level-two $A_3^{(1)}$ Verlinde fusion algebra and a ten-label commutative probability hypergroup H_{10}. In the normalized basis, each product is a probability distribution on the ten labels. An explicit bijection from the ten dominant affine weights to the ten H_{10} labels preserves quantum dimensions. Exact arithmetic in $\mathbb{Q}(\zeta_{24})$ reconstructs the fusion coefficients independently from A_3 Weyl-character values, and all one hundred ordered fusion products agree. After quantum-dimension normalization, all one hundred probability-hypergroup products agree as well.

Verlinde diagonalization, the Kac–Peterson/Weyl-character machinery, simple-current actions, and the classical Chebyshev families are standard ingredients. The contribution here is their explicit rank-ten synthesis and exact certificate: at the principal root the Weyl-character table is the Verlinde Fourier table entry-for-entry, and the canonical affine half-turn is the Galois conjugation sending the root to its fifth power, read on the column labels; the order-four simple current is affine Dynkin rotation with intrinsic $SU(4)$ four-ality phase; and reciprocal even/odd restrictions yield the first- and second-kind Chebyshev rails, while the third and fourth kinds remain rank-one shifted boundary variants rather than extra dimensions.

On the active four-point orbit, cyclic C_4 dynamics coexist with a complete $C_2 \times C_2$ binary character atlas. The resulting two-axis packet and its weighted binary Fourier traces recover all four labelled multipliers while making determinant, trace, and unordered-spectrum information loss explicit. These structures are readouts or restrictions of the ten-dimensional object, not replacement models. No Riemann-hypothesis or external analytic occupancy theorem is assumed or claimed.

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0. Exact finite theorem map and paper boundary

Reader contract. This section displays every load-bearing finite identity used in the paper. The comparison starts from two independent objects: first, H_{10}, a ten-label fusion/probability hypergroup with specified quantum dimensions and multiplication; second, the ten integrable weights of level-two $A_3^{(1)}$, evaluated by exact Weyl characters. Neither object is defined from the other. The theorem is that one explicit label bijection makes their integral multiplication laws identical; after quantum-dimension normalization, all one hundred probability products agree separately.

Names such as H-VER064 and H-AUD017 printed with the displays are internal repository provenance identifiers, not additional mathematical assumptions. They remain visible here because this is the certified Section-0 audit ledger. Appendix D records the corresponding repository paths and executable receipts.

The Chebyshev rails below are restrictions of the full ten-point Fourier calculus, not replacement rank-ten models. Later exact refinements of the active corner remain downstream provenance rather than premises of this finite reconstruction. Project-specific downstream readout comparisons are deferred to Section 10 and do not become additional premises of the certified finite graph below.

Imported sources. The following equations are owned by another paper and imported, not redrawn; each item names its owner box.

- H-PAP001 [16], box front-pap001-archimedean-pi (owner H-VER077): $\Pi_{\mathrm{src}} := \lim_{n\to\infty} P_n^- = \lim_{n\to\infty} P_n^+$ and $\Pi_{\mathrm{src}} = \pi$. Ordinary π enters only through this endpoint recognition.
- H-PAP001 [16], box front-pap001-analytic-endpoints (owner H-VER155): $\exp z := \sum_{k=0}^{\infty} z^k/k!$.

Certified owners: H-VER064 .
Covering provenance: H-VER064 .

$$
\begin{gathered}
\mathcal{B}_{10} := (1, j, \phi_3^1, \phi_3^2, \phi_1, \phi_2, \sigma_1, \sigma_2, \tau_1, \tau_2), \\
H_{10}^{\mathrm{zorb3}} = \{\mathcal{L}_{10}; *\}, \qquad |\mathcal{L}_{10}| = 10, \\
S := S_{10}^{(5.2)} = (S_{ab})_{a,b\in\mathcal{B}_{10}}, \qquad S^\dagger S = I, \\
d_a = \frac{S_{a0}}{S_{00}}, \qquad d_a > 0, \\
(d_a) = (1,1,1,1,2,2,\sqrt{3},\sqrt{3},\sqrt{3},\sqrt{3}), \\
\sum_{a\in\mathcal{L}_{10}} d_a^2 = 24, \\
\lambda_x(a) := \frac{S_{ax}}{S_{0x}}, \qquad N_a := S\ \mathrm{diag}(\lambda_x(a))_{x\in\mathcal{B}_{10}}\ S^\dagger, \\
S^\dagger N_a S = \mathrm{diag}(\lambda_x(a))_{x\in\mathcal{B}_{10}}, \\
N_a = (N_{ab}^c)_{b,c\in\mathcal{B}_{10}}, \qquad [a][b] = \sum_c N_{ab}^c [c], \\
k_a := \frac{[a]}{d_a}, \qquad k_a * k_b = \sum_c \frac{N_{ab}^c d_c}{d_a d_b}\, k_c.
\end{gathered}
\tag{1}
$$

Certified owners: H-VER067 .
Covering provenance: H-VER067 .

$$
\begin{gathered}
P_+^2(A_3) = \{(a,b,c) \in \mathbb{Z}_{\geq 0}^3 : a+b+c \leq 2\}, \qquad |P_+^2(A_3)| = 10, \\
\lambda, \mu, \nu, \tau \in P_+^2(A_3), \qquad \rho_{\mathrm{W}} = (3,2,1,0), \\
(\lambda_1, \lambda_2, \lambda_3, \lambda_4) := (a+b+c,\ b+c,\ c,\ 0) \quad \text{for} \quad \lambda = (a,b,c), \\
\text{likewise } \mu_j,\ \nu_j \text{ for } \mu, \nu, \\
\mathsf{R}(a,b,c) = (2-a-b-c, a, b), \qquad \mathsf{R}^4 = 1.
\end{gathered}
\tag{2}
$$

Certified owners: H-VER067 .
Covering provenance: H-VER067, H-VER064 .

$$\boxed{\begin{array}{c}\ell : P_+^2(A_3) \xrightarrow{\sim} \mathcal{L}_{10},\\ \begin{array}{cccccc} 1 & \leftrightarrow & (0,0,0) & j & \leftrightarrow & (0,2,0)\\ \phi_3^1 & \leftrightarrow & (2,0,0) & \phi_3^2 & \leftrightarrow & (0,0,2)\\ \phi_1 & \leftrightarrow & (0,1,0) & \phi_2 & \leftrightarrow & (1,0,1)\\ \sigma_1 & \leftrightarrow & (1,1,0) & \sigma_2 & \leftrightarrow & (0,1,1)\\ \tau_1 & \leftrightarrow & (0,0,1) & \tau_2 & \leftrightarrow & (1,0,0), \end{array}\\ d_{\ell(\lambda)} = d_\lambda,\\ O_0^{\mathrm{orb}} = (1,\phi_3^1,j,\phi_3^2), \qquad O_1^{\mathrm{orb}} = (\tau_1,\tau_2,\sigma_1,\sigma_2), \qquad O_2^{\mathrm{orb}} = (\phi_1,\phi_2),\\ 10 = 4+4+2. \end{array}} \tag{3}$$

Certified owners: H-VER068 .
Covering provenance: H-VER068, H-VER067 .

$$\boxed{\begin{array}{c}\zeta_{24} := e^{2\pi i/24}, \qquad M_j(\mu) := \mu_j + \rho_{\mathrm{W},j}, \qquad \overline{M}(\mu) := \dfrac{1}{4}\displaystyle\sum_{r=1}^{4} M_r(\mu),\\ x_{\mu,j} := \exp\Big[-\frac{2\pi i}{6}\big(M_j(\mu) - \overline{M}(\mu)\big)\Big] = \zeta_{24}^{\sum_r M_r(\mu) - 4M_j(\mu)},\\ x_\mu := (x_{\mu,j})_{j=1}^4,\\ x_{\mu,j} \in \mu_{24}. \end{array}} \tag{4}$$

Certified owners: H-VER068 .
Covering provenance: H-VER068 .

$$\boxed{\begin{array}{c} W_{\lambda\mu} := \chi_\lambda(x_\mu) = \dfrac{\det(x_{\mu,j}^{\lambda_i+\rho_{\mathrm{W},i}})}{\det(x_{\mu,j}^{\rho_{\mathrm{W},i}})},\\ W_{\lambda\mu} \in \mathbb{Q}(\zeta_{24}), \qquad \det W \neq 0. \end{array}} \tag{5}$$

Certified owners: H-VER068 .
Covering provenance: H-VER068 .

$$\boxed{\begin{array}{c} N_{\lambda\nu}^{\tau} = \Big[(W^{\mathsf{T}})^{-1}(W_{\lambda,\bullet} \odot W_{\nu,\bullet})\Big]_\tau,\\ \chi_\lambda\chi_\nu = \displaystyle\sum_{\tau \in P_+^2(A_3)} N_{\lambda\nu}^{\tau}\,\chi_\tau, \qquad \mathcal{V}_2(A_3^{(1)}) := \mathrm{span}\{\chi_\lambda\}. \end{array}} \tag{6}$$

Certified owners: H-VER068 .
Covering provenance: H-VER068, H-VER067, H-VER064 .

$$\boxed{\begin{array}{c} \Phi_{\mathrm{fus}} : \mathcal{V}_2(A_3^{(1)}) \xrightarrow{\sim} \mathcal{F}_{10}^{\mathrm{zorb3}}, \qquad \Phi_{\mathrm{fus}}(\lambda) = \ell(\lambda),\\ \Phi_{\mathrm{fus}}(\lambda\,\nu) = \Phi_{\mathrm{fus}}(\lambda)\Phi_{\mathrm{fus}}(\nu),\\ 10 \times 10 = 100 \quad \text{ordered fusion products agree exactly.} \end{array}} \tag{7}$$

Certified owners: H-VER064, H-VER068 .
Covering provenance: H-VER068, H-VER064 .

$$\begin{gathered}\kappa_\lambda := \frac{\lambda}{d_\lambda}, \qquad k_{\ell(\lambda)} = \frac{\ell(\lambda)}{d_{\ell(\lambda)}}, \\ p^\tau_{\lambda\nu} := \frac{N^\tau_{\lambda\nu}\, d_\tau}{d_\lambda d_\nu}, \qquad \Phi_H(\kappa_\lambda) = k_{\ell(\lambda)}, \\ \Phi_H(\kappa_\lambda *_W \kappa_\nu) = \Phi_H(\kappa_\lambda) *_{H_{10}^{\mathrm{zorb3}}} \Phi_H(\kappa_\nu), \\ \operatorname{Norm} \mathcal{V}_2(A_3^{(1)}) \cong H_{10}^{\mathrm{zorb3}}, \qquad 100 \quad \text{normalized products agree exactly.}\end{gathered} \tag{8}$$

Certified owners: H-VER070, H-VER064 .
Covering provenance: H-VER070, H-VER068, H-VER064 .

$$\begin{gathered}S^{(5.2)}_{10,\lambda\mu} = \frac{d_\mu}{\sqrt{24}}\, W_{\lambda\mu}, \\ \chi_\lambda(x_\mu) = \frac{S_{\lambda\mu}}{S_{0\mu}}, \qquad S_{0\mu} = \frac{d_\mu}{\sqrt{24}}, \\ \widehat{\chi}_\lambda(\mu) = \frac{\chi_\lambda(x_\mu)}{d_\lambda} = \frac{\sqrt{24}\, S_{\lambda\mu}}{d_\lambda d_\mu}, \\ \widehat{\chi}_\lambda(\mu) = \frac{1}{d_\lambda}\frac{S_{\lambda\mu}}{S_{0\mu}} = \frac{S_{\lambda\mu} S_{00}}{S_{0\mu} S_{\lambda 0}}, \\ \mathsf{R}^2(a,b,c) = (c, 2-a-b-c, a), \qquad \sigma_5(W_{\lambda,\mathsf{R}^2\mu}) = W_{\lambda\mu}, \\ \sigma_5 : \zeta_{24} \longmapsto \zeta_{24}^5.\end{gathered} \tag{9}$$

Certified owners: H-VER071 .
Covering provenance: H-VER071, H-VER070, H-VER067 .

$$\begin{gathered}g * \lambda = \mathsf{R}\lambda, \qquad q_4(a,b,c) := a + 2b + 3c \pmod 4, \\ \frac{S_{\mathsf{R}\lambda,\mu}}{S_{\lambda,\mu}} = i^{q_4(\mu)}, \qquad N_g^4 = I.\end{gathered} \tag{10}$$

Certified owners: H-VER065, H-VER069, H-VER072, H-VER082 .
Covering provenance: H-VER065, H-VER069, H-VER072, H-VER082, H-AUD017 .

$$\begin{gathered}E_n := \frac{N_g^n + N_g^{-n}}{2}, \qquad O_n := \frac{N_g^n - N_g^{-n}}{2i}, \\ E_n = T_n(E_1), \qquad O_n = O_1 U_n(E_1), \\ V_n = U_n - U_{n-1}, \qquad W_n = U_n + U_{n-1}, \\ C_{\mathrm{ch}}\, O_1\, C_{\mathrm{ch}} = -O_1.\end{gathered} \tag{11}$$

Certified owners: H-VER073 .

Covering provenance: H-VER073, H-VER071, H-VER072, H-VER074, H-VER075, H-AUD017, H-VER103, H-VER104 .

$$\boxed{\begin{gathered} X_{\mathrm{act}}^2 = J_{\mathrm{act}}^2 = E_-, \qquad X_{\mathrm{act}} J_{\mathrm{act}} = J_{\mathrm{act}} X_{\mathrm{act}}, \\ P_{\epsilon,\delta} := \frac{1}{4}(E_- + \epsilon X_{\mathrm{act}})(E_- + \delta J_{\mathrm{act}}), \\ \sum_{\epsilon,\delta=\pm 1} P_{\epsilon,\delta} = E_-, \qquad \operatorname{rank} E_- = 4. \end{gathered}} \tag{12}$$

Downstream paper boundary. The later active exponential packet and binary trace-recovery sections are retained in the manuscript as historically important finite refinements. H-VER074 and H-VER075 are the downstream theorem records, H-AUD017 is the odd-flip/fixedness audit, and H-VER103–H-VER104 are the later full-ten firewalls. They are covering provenance only and remain outside the certified dependency graph: nothing below consumes them as a premise. Appendix D records their repository paths and executable receipts.

Certified finite package.

Certified owners: H-VER064, H-VER065, H-VER067, H-VER068, H-VER069, H-VER070, H-VER071, H-VER072, H-VER073, H-VER082 .

Covering provenance: H-VER064, H-VER065, H-VER067, H-VER068, H-VER069, H-VER070, H-VER071, H-VER072, H-VER073, H-VER082, H-VER074, H-VER075, H-AUD017, H-VER103, H-VER104 .

$$\boxed{\begin{gathered} \mathcal{V}_2(A_3^{(1)}) \cong \mathcal{F}_{10}^{\mathrm{zorb3}}, \qquad \operatorname{Norm} \mathcal{V}_2(A_3^{(1)}) \cong H_{10}^{\mathrm{zorb3}}, \\ \text{exact cyclotomic reconstruction, Fourier identification,} \\ \text{Chebyshev restrictions and active binary readouts are finite,} \\ \text{with no external analytic source, zeta, Xi or RH premise/conclusion.} \end{gathered}} \tag{13}$$

1 The level-two A3 alcove

Write a finite A_3 dominant weight as $\lambda = (a, b, c)$. The integrable level-two alcove is

$$P_+^2(A_3) = \{(a, b, c) \in \mathbb{Z}_{\geq 0}^3 : a + b + c \leq 2\}.$$

It contains

$$\binom{2+3}{3} = 10$$

weights. The affine Dynkin labels are

$$(2 - a - b - c, a, b, c).$$

The order-four affine rotation acts in finite-label coordinates by

$$\mathsf{R}(a, b, c) = (2 - a - b - c, a, b).$$

Contragredient duality is

$$(a, b, c)^* = (c, b, a).$$

The exact label dictionary is the visible ten-sector naming used in the earlier D12RG construction and its connected Verlinde completion [17, 18, 19]:

1	$(0,0,0)$	j	$(0,2,0)$
g	$(2,0,0)$	$\bar{g}$	$(0,0,2)$
ϕ_1	$(0,1,0)$	ϕ_2	$(1,0,1)$
σ_1	$(1,1,0)$	σ_2	$(0,1,1)$
τ_1	$(0,0,1)$	τ_2	$(1,0,0)$

The simple-current orbits are

$$1 \to g \to j \to \bar{g} \to 1,$$

$$\tau_1 \to \tau_2 \to \sigma_1 \to \sigma_2 \to \tau_1,$$

and

$$\phi_1 \leftrightarrow \phi_2.$$

The Weyl quantum-dimension formula at level two gives exactly the multiset $1^4, 2^2, (\sqrt{3})^4$.

2 Exact Weyl-character reconstruction

For $A_3 \cong \mathfrak{sl}_4$, write the partition associated to $\lambda = (a, b, c)$ as

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (a + b + c, b + c, c, 0).$$

Let

$$\rho = (3, 2, 1, 0), \qquad k + h^\vee = 2 + 4 = 6.$$

For a spectral weight $\mu = (r, s, t)$ define its partition coordinates by

$$m(\mu) = (r + s + t, s + t, t, 0),$$

and set

$$M_j(\mu) = m_j(\mu) + \rho_j, \qquad T(\mu) = \sum_{j=1}^{4} M_j(\mu), \qquad \overline{M}(\mu) = \frac{T(\mu)}{4}.$$

Centering in $SU(4)$ gives

$$\boxed{x_{\mu,j} = \exp\left[-\frac{2\pi i}{6}(M_j(\mu) - \overline{M}(\mu))\right] = \zeta_{24}^{T(\mu)-4M_j(\mu)}, \qquad \zeta_{24} = e^{2\pi i/24}.}$$

The exponent $T(\mu) - 4M_j(\mu)$ is integral, so every coordinate lies in μ_{24}. The centering also gives

$$\prod_{j=1}^{4} x_{\mu,j} = 1,$$

so x_μ is an $SU(4)$ torus point rather than merely a four-tuple of roots of unity. This is the exact reason that the character calculation closes in the cyclotomic field $\mathbb{Q}(\zeta_{24})$.

The Weyl character formula is evaluated as

$$\chi_\lambda(x_\mu) = \frac{\det(x_{\mu,j}^{\lambda_i+\rho_i})_{i,j=1}^{4}}{\det(x_{\mu,j}^{\rho_i})_{i,j=1}^{4}}.$$

Write

$$W_{\lambda\mu} = \chi_\lambda(x_\mu).$$

The use of Weyl-character values at shifted finite-order torus points is standard in the Kac–Peterson description of affine modular data: the character ratio $S_{\lambda\mu}/S_{0\mu}$ is precisely the finite Weyl-character eigenvalue used by the Verlinde algebra [2, 1, 11]. The paper-specific question is therefore not whether such a character realization exists, but whether this exact ten-point cyclotomic table reconstructs the independently supplied rank-ten source multiplication. The resulting ten-by-ten table is invertible over $\mathbb{Q}(\zeta_{24})$. Hence pointwise multiplication of two rows reconstructs the fusion coefficients exactly:

$$\boxed{N_{\lambda\nu}^{\tau} = \Big[(W^{\mathsf{T}})^{-1}(W_{\lambda,\bullet} \odot W_{\nu,\bullet})\Big]_{\tau}.}$$

No floating-point reconstruction is used. The source rank-ten fusion table is the connected Verlinde datum of Paper 5.2 and its SM-008 proof supplement [18, 19]. It enters only after this independent calculation, at the final comparison boundary of Section 3.

Exact theorem provenance and the executable reconstruction receipt are listed in Appendix D.

3 The exact rank-ten isomorphism

Theorem 3.1 (Weyl–Verlinde fusion isomorphism)**.** *Under the dictionary of Section 1, exact Weyl-character reconstruction in $\mathbb{Q}(\zeta_{24})$ gives the same multiplication as the rank-ten Verlinde fusion algebra for all one hundred ordered products [17, 18, 19]. Let $\ell : P_+^2(A_3) \to \mathcal{L}_{10}$ denote the ten-label bijection of Section 1. On the integral fusion bases the isomorphism is therefore the explicit map*

$$\Phi_F : [\lambda] \longmapsto [\ell(\lambda)], \qquad \Phi_F^{-1} : [a] \longmapsto [\ell^{-1}(a)].$$

Hence

$$\boxed{\mathcal{V}_2(A_3^{(1)}) \cong \mathcal{F}_{10}.}$$

The comparison is stronger than agreement of rank or dimension data. The executable owner compares every coefficient of every ordered fusion product. Commutativity reduces a printed presentation to fifty-five independent products; these are given in Section 4.

The result is compatible with the known fusion-algebra identification of the rank-one lattice orbifold with affine type $D^{(1)}$ at level two [7]. For $D_3 \cong A_3$, this gives the same affine fusion type. More generally, affine fusion-ring automorphisms and isomorphic pairs have been classified, and polynomial realizations of type-A fusion algebras predate the present calculation [12, 11]. The contribution here is therefore not the existence of the affine fusion type in isolation. It is the explicit ten-label identification with the independently supplied source system, together with the exact comparison certificates used below.

4 Complete fusion algebra

The identity law is

$$1\, a = a \qquad (a \in \mathcal{L}).$$

Together with commutativity, the following forty-five nontrivial products complete the fifty-five independent products of the algebra.

Products with j

$$\begin{aligned} j^2 &= 1, & jg &= \bar{g}, & j\bar{g} &= g, \\ j\phi_1 &= \phi_1, & j\phi_2 &= \phi_2, & j\sigma_1 &= \tau_1, \\ j\sigma_2 &= \tau_2, & j\tau_1 &= \sigma_1, & j\tau_2 &= \sigma_2. \end{aligned}$$

Products with g and $\bar{g}$

$$
\begin{aligned}
g^2 &= j, & g\bar{g} &= 1, & g\phi_1 &= \phi_2,\\
g\phi_2 &= \phi_1, & g\sigma_1 &= \sigma_2, & g\sigma_2 &= \tau_1,\\
g\tau_1 &= \tau_2, & g\tau_2 &= \sigma_1, & \bar{g}^2 &= j,\\
\bar{g}\phi_1 &= \phi_2, & \bar{g}\phi_2 &= \phi_1, & \bar{g}\sigma_1 &= \tau_2,\\
\bar{g}\sigma_2 &= \sigma_1, & \bar{g}\tau_1 &= \sigma_2, & \bar{g}\tau_2 &= \tau_1.
\end{aligned}
$$

Products with ϕ_1 and ϕ_2

$$
\begin{aligned}
\phi_1^2 &= 1 + j + \phi_2, & \phi_1\phi_2 &= g + \bar{g} + \phi_1,\\
\phi_1\sigma_1 &= \sigma_2 + \tau_2, & \phi_1\sigma_2 &= \sigma_1 + \tau_1,\\
\phi_1\tau_1 &= \sigma_2 + \tau_2, & \phi_1\tau_2 &= \sigma_1 + \tau_1,\\
\phi_2^2 &= 1 + j + \phi_2, & \phi_2\sigma_1 &= \sigma_1 + \tau_1,\\
\phi_2\sigma_2 &= \sigma_2 + \tau_2, & \phi_2\tau_1 &= \sigma_1 + \tau_1,\\
\phi_2\tau_2 &= \sigma_2 + \tau_2.
\end{aligned}
$$

Products inside the active four-state sector

$$
\begin{aligned}
\sigma_1^2 &= \bar{g} + \phi_1, & \sigma_1\sigma_2 &= 1 + \phi_2,\\
\sigma_1\tau_1 &= g + \phi_1, & \sigma_1\tau_2 &= j + \phi_2,\\
\sigma_2^2 &= g + \phi_1, & \sigma_2\tau_1 &= j + \phi_2,\\
\sigma_2\tau_2 &= \bar{g} + \phi_1, & \tau_1^2 &= \bar{g} + \phi_1,\\
\tau_1\tau_2 &= 1 + \phi_2, & \tau_2^2 &= g + \phi_1.
\end{aligned}
$$

5 The normalized finite hypergroup

The connected rank-ten source fusion datum and its typed transport are the public-paper ancestry of this normalization [18, 19]. Finite commutative hypergroups and fusion rule algebras already have a common standard framework [8]; in particular, normalization by the positive dimension character is not claimed here as a new construction. Let N_{ab}^{c} denote the fusion coefficients and let d_a be the positive quantum dimension. Define the normalized basis

$$k_a = \frac{a}{d_a}.$$

Then

$$\boxed{k_a * k_b = \sum_c \frac{N_{ab}^{c} d_c}{d_a d_b}\, k_c.}$$

Every coefficient is nonnegative and the coefficients sum to one. The involution is fusion duality, and the identity occurs in $k_a * k_b$ exactly when $b = a^*$.

The Haar probability is

$$\boxed{h(a) = \frac{d_a^2}{24}.}$$

In source order this is

$$\frac{1}{24}(1,1,1,1,4,4,3,3,3,3).$$

The executable theorem checks identity, commutativity, all one thousand associativity triples, involution, positivity, normalization, and Haar invariance.

Theorem 5.1 (Normalized hypergroup isomorphism)**.** *Let $\ell : P_+^2(A_3) \to \mathcal{L}_{10}$ be the label dictionary and write*

$$\kappa_\lambda = \frac{[\lambda]}{d_\lambda}, \qquad k_a = \frac{[a]}{d_a}.$$

The dimension match $d_{\ell(\lambda)} = d_\lambda$ makes

$$\boxed{\Phi_H(\kappa_\lambda) = k_{\ell(\lambda)}, \qquad \Phi_H^{-1}(k_a) = \kappa_{\ell^{-1}(a)}}$$

a bijection of the two normalized bases. If

$$p_{\lambda\mu}^{(A_3)\nu} = \frac{N_{\lambda\mu}^{\ \nu} d_\nu}{d_\lambda d_\mu},$$

then the executable comparison of all one hundred ordered products gives

$$p_{\lambda\mu}^{(A_3)\nu} = p_{\ell(\lambda)\ell(\mu)}^{(H_{10})\ \ell(\nu)}.$$

Consequently

$$\boxed{\Phi_H(\kappa_\lambda *_{A_3} \kappa_\mu) = \Phi_H(\kappa_\lambda) *_{H_{10}} \Phi_H(\kappa_\mu), \qquad \operatorname{Norm} \mathcal{V}_2(A_3^{(1)}) \cong H_{10}.}$$

Thus the integral fusion bases and normalized stochastic bases are separately typed, while the normalized Weyl/Verlinde and rank-ten probability-hypergroup presentations are proved isomorphic rather than merely associated.

The normalized character table is symmetric and invertible. Its dual hypergroup reconstructs all one hundred original normalized products, so the finite hypergroup is strongly self-dual.

6 Weyl characters and the Verlinde Fourier table

Let S denote the modular matrix and let R denote the affine order-four Dynkin rotation. With the principal root ζ_{24} of Section 2, the source columns and the Weyl points carry the same labels: no relabelling is needed.

Theorem 6.1 (Exact Fourier equality)**.** *For all one hundred pairs of level-two weights,*

$$\boxed{\chi_\lambda^{A_3}(\mu) = \frac{S_{\lambda\mu}}{S_{0\mu}}.}$$

Moreover

$$S_{0\mu} = \frac{d_\mu}{\sqrt{24}}.$$

After division of row λ by d_λ, the same table is the normalized probability-hypergroup character table.

Corollary 6.2 (Galois conjugate embedding)**.** *Let* $\sigma_5 \in \mathrm{Gal}(\mathbb{Q}(\zeta_{24})/\mathbb{Q})$ *send* $\zeta_{24} \mapsto \zeta_{24}^5$*, and let* $W_{\lambda\mu}$ *be the Weyl table of Section 2. Then, for all one hundred pairs,*

$$\boxed{\sigma_5(W_{\lambda,\mathsf{R}^2\mu}) = W_{\lambda\mu}.}$$

The Galois action σ_5 *appears on the column labels as the affine half-turn* R^2*. Since* $5 \cdot 5 \equiv 1 \pmod{24}$*,* σ_5 *is an involution. An exact computation in a cyclotomic field whose generator embeds as* ζ_{24}^5 *sees the table in this form.*

The general relation between discretized Weyl-orbit transforms and affine modular data is established in the literature [11, 13, 14]. Thus the standard ingredient here is the Weyl/Kac–Peterson transform itself. The paper-specific finite statement is that the supplied source Fourier table agrees entry-by-entry with the reconstructed table at the principal root with no relabelling, the half-turn $\mu \mapsto \mathsf{R}^2\mu$ being the trace of the Galois conjugation σ_5, and that the same normalized table is the character table of the proved hypergroup isomorphism.

7 Simple current and four-ality

Simple-current actions and their modular phase laws are standard in affine fusion theory [9, 11]. In the present label convention the order-four current is

$$g \leftrightarrow (2, 0, 0).$$

Fusion by g is exactly affine Dynkin rotation:

$$\boxed{g \star \lambda = \mathsf{R}\lambda.}$$

Define the intrinsic $SU(4)$ four-ality by

$$q(a, b, c) = a + 2b + 3c \pmod 4.$$

Then the simple-current Fourier character is

$$\boxed{\chi_g(\lambda) = i^{q(\lambda)}.}$$

All one hundred covariance identities obey the corresponding phase rule. Contragredient duality negates four-ality.

The ten weights therefore carry a canonical map to $\mathbb{Z}/4\mathbb{Z}$. The fibres of that map are not the complete Weyl labels; they are the rank-one charge coordinate on which the Chebyshev restriction acts.

8 Classical and generalized Chebyshev structure

Generalized Chebyshev polynomials associated with Weyl and affine-Weyl symmetry are classical, and later orbit-function work makes the multivariate Lie-theoretic character connection explicit [4, 5, 13, 15]. These general constructions are not claimed as new here. The issue below is how their rank-one restrictions sit inside the already identified ten-point fusion object.

Notation firewall. We reserve R for affine Dynkin rotation and g for the order-four simple current. The reciprocal even/odd rails are denoted E_n, O_n, while T_n, U_n, V_n, W_n retain their standard meanings as the four classical Chebyshev families. The later active binary operators are written $X_{\mathrm{act}}, J_{\mathrm{act}}$; lower-case j remains the involutive fusion label of Section 1.

For the reciprocal simple-current pair define

$$E_n = \frac{N_g^n + N_g^{-n}}{2}, \qquad O_n = \frac{N_g^n - N_g^{-n}}{2i}.$$

Then

$$\boxed{E_n = T_n(E_1),}$$

and

$$\boxed{O_n = O_1 U_{n-1}(E_1).}$$

Equivalently,

$$N_g^n = E_n + iO_n.$$

The first-kind product law is the familiar inversion-orbit hypergroup law

$$E_m E_n = \frac{1}{2}\left(E_{m+n} + E_{|m-n|}\right).$$

Since $g^4 = 1$, radialization by $q \sim -q$ reduces the charge coordinate to three classes. This is a rank-one shadow, not the full rank-ten hypergroup.

The reciprocal trace/Chebyshev strand also appears in the later D12RG Paper 6.2 and its trace–determinant generalization in Paper 6.9 [21, 24]. Those papers are project-lineage context, not inputs to the finite reconstruction proved here.

The four classical kinds share one recurrence. In standard normalization,

$$V_n = U_n - U_{n-1}, \qquad W_n = U_n + U_{n-1},$$

while their half-angle forms exhibit the shifted boundary conditions. Thus T, U, V, W model the rank-one orbit/sign/half-shift grammar. The full object is the higher-rank A_3 Weyl-character quotient.

The third/fourth-kind boundary also retains an exact operator doublet. The current certified theorem instantiates the V/W half-step doublet in the rank-ten Verlinde carrier. With the oriented odd-rail operator O_1 and charge conjugation C, its odd-flip relation is

$$CO_1C = -O_1.$$

An earlier oriented-lift note and a later odd-flip audit remain useful historical provenance, but they are not load-bearing premises of this paper statement; their exact repository identities are listed in Appendix D. Polynomial functional calculus then gives

$$CV_n(O_1)C = (-1)^n W_n(O_1), \qquad CW_n(O_1)C = (-1)^n V_n(O_1),$$

and

$$V_n(O_1)W_n(O_1) = U_{2n}(O_1).$$

This is a rank-one half-step doublet inside the full A_3 algebra; it is not an additional independent rank-ten Weyl basis.

9 The active orbit: cyclic and binary structure

The active four-point orbit is

$$\tau_1 \to \tau_2 \to \sigma_1 \to \sigma_2 \to \tau_1.$$

Here lower-case j is the involutive fusion label of Section 1, not an operator symbol. Let

$$E_- = \frac{I - N_j}{2}.$$

Define

$$\boxed{X_{\mathrm{act}} = \frac{\sqrt{2}}{4}(-N_{\sigma_1} - N_{\sigma_2} + N_{\tau_1} + N_{\tau_2})}$$

and

$$\boxed{J_{\mathrm{act}} = \frac{N_{\bar{g}} - N_g}{2i} = -O_1.}$$

Their Fourier values on the ordered active orbit are

$$X_{\mathrm{act}} = (+,+,-,-), \qquad J_{\mathrm{act}} = (+,-,+,-).$$

They obey

$$\boxed{X_{\mathrm{act}}^2 = J_{\mathrm{act}}^2 = E_-, \qquad X_{\mathrm{act}} J_{\mathrm{act}} = J_{\mathrm{act}} X_{\mathrm{act}}.}$$

Hence the simple-current motion has cyclic dynamics C_4, while the two commuting signs form a complete binary $C_2 \times C_2$ character atlas. The four rank-one projectors are

$$\boxed{P_{\epsilon,\delta} = \frac{1}{4}(E_- + \epsilon X_{\mathrm{act}})(E_- + \delta J_{\mathrm{act}}).}$$

They are mutually orthogonal and sum to E_-.

An older indexed triangular source retained

$$T_-(n) = \frac{n(n-1)}{2}, \qquad n.$$

For $n \bmod 4 = 0, 1, 2, 3$,

$$(-1)^{T_-(n)} = (+,+,-,-), \qquad (-1)^n = (+,-,+,-).$$

Thus those two unsquared source bits are exactly the two binary coordinates of the active Weyl orbit.

9.1 Historical role and downstream refinements

The $C_2 \times C_2$ atlas above is the paper's original active-corner coordinate theorem: it resolves the four active spectral characters and keeps the two unsquared parity bits distinct. Later work should not be read backwards as a premise of this construction.

Subsequent exact work exponentiated the same two commuting axes and classified which scalar readouts lose orientation. Those results are retained in Appendix B as downstream refinements. A later audit clarified that charge conjugation, not X_{act}, is the exact odd flip on J_{act}, and that Reynolds fixedness averages away the odd coordinate rather than selecting it. Repository identifiers for these results are collected in Appendix D.

Still later full-ten work supplies an important typing firewall. A supported rank-four corner remains a corner of literal 10×10 operators, and the four active coordinate labels do not identify the coordinate seed with the rank-four spectral support. Thus the active atlas is a precise witness and readout sector of the rank-ten Weyl/Verlinde object, not a replacement model for that object.

10 Information loss, theorem boundary, and reproducibility

The constructions distinguish a full coordinate from a readout that forgets orientation. In particular,

$$J_{\mathrm{act}} \longmapsto J_{\mathrm{act}}^2 = E_-$$

retains active support but identifies J_{act} and $-J_{\mathrm{act}}$. Likewise the first-kind Chebyshev radialization retains the reciprocal sum and loses the odd rail.

The higher-rank Weyl/Verlinde isomorphism does not repair such a loss by itself. It supplies a canonical full-rank coordinate system in which one can state precisely which functions have been retained.

All finite identities in this paper have exact executable receipts. The A_3 Weyl reconstruction is exact in $\mathbb{Q}(\zeta_{24})$; all one hundred fusion products and all one hundred Fourier identities are checked symbolically. The active-orbit projector theorem, two-axis exponential packet, and binary weighted-trace recovery are also checked exactly. Repository theorem identifiers, audit ancestry, implementation paths, and reproduction commands are collected in Appendix D.

For provenance, three mathematical levels must be kept distinct: the certified finite reconstruction spine, the current theorem supporting the rank-ten V/W operator doublet, and later theorem/audit refinements of the active corner. The last group records consistency and information-loss clarifications but does not enter the reconstruction as an upstream premise.

Standard literature and contribution boundary. The Verlinde diagonalization, Kac–Peterson modular matrix, Weyl-character eigenvalue formula, simple-current action, fusion-rule hypergroup framework, and generalized Chebyshev/Weyl machinery are all standard ingredients [1, 2, 11, 8, 9, 4, 14]. The contribution claimed here is their explicit finite synthesis on the particular rank-ten surface studied above: a ten-label ${A_3^{(1)}}_2 \leftrightarrow H_{10}$ dictionary, an exact $\mathbb{Q}(\zeta_{24})$ Weyl reconstruction, complete one-hundred-product certificates for both the integral fusion law and its normalized probability-hypergroup law, the entrywise identification of the supplied Fourier table at the principal root together with its σ_5 Galois-conjugate reading through the R^2 relabelling, and the placement of the Chebyshev and $C_2 \times C_2$ readouts as restrictions of that same rank-ten object. We do not claim that any one of those standard ingredients is new in isolation, nor do we claim a general new construction of affine fusion rings or generalized Chebyshev polynomials.

Programme role: zorb3, D12RG, and GKS. The source direction remains one-way. The connected rank-ten zorb3/Verlinde probability hypergroup is established independently of any Gang–Kim–Stubbs (GKS) comparison. The present paper supplies an exact A_3 Weyl/Verlinde/generalized-Chebyshev realization of that full-rank finite owner, and therefore a finite witness/readout surface which later D12RG/GKS comparisons may consume. They do not source or calibrate it.

The relevant finite-source theorem makes the information-loss direction explicit on the Fourier side: the D12RG full Fourier/half-character source retains phase and orientation before an inversion-odd fold produces the sine-shaped GKS readout. That fold is lossy and is not inverted to manufacture omitted source coordinates. Its repository identity is given in Appendix D. Paper 6.5 imports the U_Φ carrier context of Paper 6.4 and uses the same guarded representational viewpoint at the cyclotomic, exponential, sine, and cosine layer [22, 23]. It preserves the non-identification boundary $\mathcal{GKS} \neq \mathcal{D}\infty\in\mathcal{RG}$. Consequently the present paper supplies an exact rank-ten finite comparison surface for those later readouts; it does not identify GKS with D12RG and does not import GKS as an upstream premise. Paper 5.3 owns a different finite task—admissible quotient/readout reconstruction—and is cited here

precisely to prevent that readout reconstruction from being mistaken for the connected source fusion algebra [20].

No external analytic source is assigned to a Weyl point by this paper. No proof of the Riemann hypothesis is claimed.

A A worked fusion product and exhaustive verification

This appendix works one nontrivial product through explicitly and then points to the exact executable receipt for the complete rank-ten table. The purpose is not to replace the all-product verification by a single example. It is to make the reconstruction mechanism visible to a reader before the exhaustive exact check is invoked. This is an explanatory consequence of the exact reconstruction theorem and introduces no additional mathematical premise.

A.1 The product $\phi_1 * \phi_1$

Under the label dictionary of Section 1,

$$\phi_1 \longleftrightarrow (0,1,0), \qquad j \longleftrightarrow (0,2,0), \qquad \phi_2 \longleftrightarrow (1,0,1), \qquad 1 \longleftrightarrow (0,0,0).$$

The partition map used in Section 2 sends

$$(a,b,c) \longmapsto (a+b+c, b+c, c, 0),$$

so the character of ϕ_1 is the $SU(4)$ Schur character $s_{(1,1)}$. The Littlewood–Richardson product is

$$s_{(1,1)}^2 = s_{(2,2)} + s_{(2,1,1)} + s_{(1,1,1,1)}.$$

On $SU(4)$, the last term is the determinant character and therefore is equal to one. Converting the remaining partitions back to finite Dynkin labels gives

$$(2,2,0,0) \longleftrightarrow (0,2,0), \qquad (2,1,1,0) \longleftrightarrow (1,0,1).$$

All three resulting weights lie in the level-two alcove. Hence this particular product requires no further affine truncation, and

$$\boxed{[\phi_1][\phi_1] = [1] + [j] + [\phi_2].} \tag{14}$$

The same identity is seen directly by the exact Weyl table. If $W_{a,\bullet}$ denotes the row of character evaluations belonging to label a, then pointwise multiplication gives

$$W_{\phi_1,\bullet} \odot W_{\phi_1,\bullet} = W_{1,\bullet} + W_{j,\bullet} + W_{\phi_2,\bullet}. \tag{15}$$

Applying the exact inverse transpose $(W^{\mathsf{T}})^{-1}$ therefore returns coefficient one on precisely those three basis elements and zero on the remaining seven. This is the same reconstruction operation used for every ordered pair in the exhaustive exact certificate.

A.2 The normalized probability-hypergroup product

The quantum dimensions are

$$d_{\phi_1} = 2, \qquad d_1 = d_j = 1, \qquad d_{\phi_2} = 2.$$

With $k_a = [a]/d_a$, equation (14) therefore becomes

$$\boxed{k_{\phi_1} * k_{\phi_1} = \frac{1}{4}k_1 + \frac{1}{4}k_j + \frac{1}{2}k_{\phi_2}.} \tag{16}$$

The coefficients are nonnegative and sum to one. Thus the same worked example displays both the integral fusion product and its normalized probability-hypergroup image.

A.3 Exact verification of the complete table

The hand calculation above is only an explanatory slice. The complete certificate independently reconstructs every ordered product from the exact cyclotomic Weyl-character table before comparing with the rank-ten source hypergroup. It requires exactly one hundred ordered fusion products to match. A second exact certificate independently requires all one hundred normalized products to match and thereby certifies the full probability-hypergroup isomorphism. No floating-point reconstruction is used in either calculation.

Implementation paths and the direct reproduction command are collected in Appendix D, keeping repository mechanics out of the worked mathematical example.

B Downstream exact refinements of the active $C_2 \times C_2$ corner

Provenance boundary. This section records exact downstream refinements of the active corner. They are retained as later structure, not as premises of the finite reconstruction. Their repository theorem and audit identifiers are listed in Appendix D.

The active atlas admits a canonical two-parameter exponential on the active corner:

$$\boxed{M(\eta,\theta) = \exp(\eta J_{\mathrm{act}} - i\theta X_{\mathrm{act}}), \qquad \eta,\theta \in \mathbb{R}.}$$

Since the joint projectors satisfy

$$X_{\mathrm{act}} P_{\epsilon,\delta} = \epsilon P_{\epsilon,\delta}, \qquad J_{\mathrm{act}} P_{\epsilon,\delta} = \delta P_{\epsilon,\delta},$$

one has the exact projector expansion

$$\boxed{M(\eta,\theta) = \sum_{\epsilon,\delta=\pm 1} e^{\delta\eta - i\epsilon\theta} P_{\epsilon,\delta}.}$$

Thus the four labeled multipliers are

$$e^{+\eta - i\theta}, \quad e^{-\eta - i\theta}, \quad e^{+\eta + i\theta}, \quad e^{-\eta + i\theta}.$$

The corner identity is E_- and

$$M(\eta_1,\theta_1) M(\eta_2,\theta_2) = M(\eta_1+\eta_2, \theta_1+\theta_2).$$

In particular,

$$\boxed{M(\eta,\theta)^{-1}_{E_-} = M(-\eta,-\theta).}$$

This is a corner inverse: the supported 10×10 matrix is singular on the complementary six-dimensional sector.

Let

$$\Theta(A) = C\overline{A}C.$$

Although $CJ_{\mathrm{act}}C = -J_{\mathrm{act}}$, complex conjugation also sends J_{act} to $-J_{\mathrm{act}}$; hence

$$\Theta(X_{\mathrm{act}}) = X_{\mathrm{act}}, \qquad \Theta(J_{\mathrm{act}}) = J_{\mathrm{act}}.$$

Therefore

$$\boxed{\Theta(M) = M^\dagger = M(\eta,-\theta).}$$

Comparison with the corner inverse gives

$$\boxed{\Theta(M) = M^{-1}_{E_-} \quad \Longleftrightarrow \quad \eta = 0.}$$

The difference factors through $\sinh\eta$ on every joint character.

The same projector basis gives a sharp information-loss hierarchy. The corner determinant is

$$\boxed{\det_{E_-} M = 1}$$

for all η, θ. The four binary Fourier traces are

$$\begin{aligned}
T_{00} &= \mathrm{tr}_{E_-}(M) = 4\cosh\eta\cos\theta, \\
T_{10} &= \mathrm{tr}_{E_-}(X_{\mathrm{act}}M) = -4i\cosh\eta\sin\theta, \\
T_{01} &= \mathrm{tr}_{E_-}(J_{\mathrm{act}}M) = 4\sinh\eta\cos\theta, \\
T_{11} &= \mathrm{tr}_{E_-}(X_{\mathrm{act}}J_{\mathrm{act}}M) = -4i\sinh\eta\sin\theta.
\end{aligned}$$

They recover every labeled multiplier by the inverse $C_2 \times C_2$ transform

$$\boxed{m_{\epsilon,\delta} = \frac{1}{4}\left(T_{00} + \epsilon T_{10} + \delta T_{01} + \epsilon\delta T_{11}\right).}$$

Thus determinant loses the radial coordinate entirely; ordinary trace and the unordered spectrum lose its sign; the J_{act}-weighted trace retains that orientation. If $\cos\theta \neq 0$, then

$$\mathrm{tr}_{E_-}(J_{\mathrm{act}}M) = 0 \quad \Longleftrightarrow \quad \eta = 0.$$

The nonzero-cosine condition is essential.

B.1 Later audit and full-ten firewalls

A later audit distinguishes linear charge transport from the anti-linear involution used above. Charge conjugation fixes X_{act} and reverses J_{act}, so it sends $M(\eta,\theta)$ to $M(-\eta,\theta)$. Pointwise charge fixedness is therefore an endpoint equalizer at $\eta = 0$; the charge-Reynolds average is fixed for every η only because it discards the odd coordinate. This is a downstream information-loss clarification, not an additional source law.

Two superficially similar $C_2 \times C_2$ appearances must also remain typed separately. One is the generic commuting-involution/Hadamard-square calculus; the present paper uses the specific faithful active-Weyl realization. A different downstream DVJ source problem constructs a signed-root reciprocal doublet. Sharing the same abstract binary character table does not identify those source objects with the active orbit here.

Still later work returns to literal full-ten operators. It shows, respectively, that supported rank-four Pauli corners remain 10×10 supported operator statements and that the four active coordinate labels generate a seed which is *not* equal to the rank-four spectral support. These later theorems confirm the interpretation used throughout this paper:

$$\boxed{\text{active four-point corner } \neq \text{ replacement for the rank-ten semantic carrier.}}$$

They are downstream consistency checks only; none is back-propagated as a premise of the active atlas or of the Weyl/Verlinde reconstruction.

C D12RG prior-work citation and source ledger

This appendix records the internal publication ancestry used by the paper. The entries were checked against the canonical project LaTeX or coordinator sources rather than reconstructed from paper numbers alone. Their role is bibliographic and provenance-facing. They add no mathematical premise to the finite reconstruction.

Paper 5: hidden-core and visible-readout completion. Paper 5 is the upstream D12RG CFT completion paper and fixes the hidden-core/visible-readout ten-sector setting used by the later sequence [17]. It is prior-work ancestry for the visible label system, not the independent A_3 Weyl reconstruction performed here.

Paper 5.2 and SM-008: connected Verlinde source. Paper 5.2 owns the connected ten-sector source modular datum, ordinary source-label Verlinde calculation, and typed transport to the Paper 5 visible/readout labels; SM-008 is its proof supplement [18, 19]. This is the immediate public-paper ancestry of the rank-ten source fusion table compared with the independent Weyl reconstruction in Sections 2 and 3.

Paper 5.3: quotient and readout reconstruction. Paper 5.3 owns the admissible scalar/matrix reconstruction on the finite quotient/readout side, including the typed reconstruction identities. It does not replace the connected source modular datum of Paper 5.2 and is therefore not used as the source of the rank-ten fusion algebra in this paper [20].

Paper 6.2: reciprocal closure and Chebyshev context. Paper 6.2 develops the reciprocal product/trace closure strand and the cyclotomic branch grammar later reused throughout the D12RG sequence [21]. It is cited here as prior project context for the reciprocal Chebyshev language of Section 8, not as a premise of the rank-ten Weyl/Verlinde isomorphism.

Paper 6.4: the U_Φ carrier. Paper 6.4 isolates the golden modular carrier, its Dickson–Lucas–Chebyshev trace ladder, and normalized lift closure [22]. This is downstream carrier context for the later GKS comparison, not an upstream input to the present reconstruction.

Paper 6.5: guarded GKS/D12RG comparison. Paper 6.5 is the downstream source-to-modular/cyclotomic comparison paper [23]. It states the boundary

$$\boxed{\mathcal{GKS} \neq \mathcal{D}\infty\in\mathcal{RG}}$$

and instead compares the GKS sine kernel and D12RG/zorb3 witness as readouts of a common cyclotomic trace grammar. The present paper supplies a finite rank-ten comparison surface to that later programme; Paper 6.5 does not source the present reconstruction.

Paper 6.9: trace–determinant generalization. Paper 6.9 develops the general trace–determinant companion-matrix family and its Dickson–Lucas–Chebyshev structure [24]. It is a later generalization neighboring the rank-one Chebyshev language of this paper, not a source of the rank-ten object.

Paper 7.4: downstream DVJ radial readout. Paper 7.4 owns a later DVJ/golden-spiral radial readout [25]. It is recorded for sequence completeness only; no Paper 7.4 theorem is used in the finite Weyl/Verlinde reconstruction.

Typed lineage summary. The publication ancestry used here is therefore

Paper 5: hidden-core/visible-readout setting,
Paper 5.2 + SM-008: connected source Verlinde datum,
Paper 5.3: quotient/readout reconstruction,
Papers 6.2/6.9: reciprocal/Chebyshev lineage,
Papers 6.4/6.5: downstream carrier and GKS comparison,
Paper 7.4: downstream DVJ/golden-spiral readout.

Only the first two lines are immediate ancestry of the rank-ten source fusion datum compared here. The remaining lines preserve project context and direction without becoming hidden proof ingress.

D Repository provenance and executable receipts

This appendix records development provenance without making repository identifiers prerequisites for reading the mathematical body. The public repository containing the theorem notes, audits, exact implementations, and paper sources is `https://gitlab.com/d12rg/d12rg-hypergroup`. The identifiers below are provenance labels rather than extra mathematical assumptions.

Reader-notation translation. The publication reserves R for affine Dynkin rotation, O_n for the odd reciprocal Chebyshev rail, and $X_{\mathrm{act}}, J_{\mathrm{act}}$ for the active binary operators. Older repository notes may use J for the affine rotation, J_n for the odd rail, or X, J for the active pair. These are presentation aliases only; the source labels, operators, and exact executable maps are unchanged.

D.1 Certified finite reconstruction spine

H-VER064. The connected rank-ten fusion and probability-hypergroup source is recorded in

`docs/theorems/h-ver064-rank-ten-zorb3-verlinde-hypergroup.md`.

H-VER065 and H-VER067–H-VER073. These closed theorem notes own, respectively, the Chebyshev/Verlinde orbit embedding, the independent $A_3^{(1)}$ level-two alcove and dictionary, the exact cyclotomic reconstruction and fusion/hypergroup isomorphism, the four-kind Chebyshev boundary, the Weyl/Verlinde Fourier identification, the simple-current four-ality law, the Chebyshev restriction of the full Fourier calculus, and the active $C_2 \times C_2$ atlas. Their exact source paths are frozen in `support/source-manifest.json` for this paper.

H-VER082. The current certified source of the rank-ten third/fourth-kind V/W half-step operator statement used in Section 8 is

`docs/theorems/h-ver082-chebyshev-vw-halfstep-doublet.md`.

The earlier H-VER066 oriented-lift note is historical provenance rather than its current owner.

D.2 Downstream refinement and audit provenance

H-VER074, H-VER075 and H-AUD017. The closed downstream records are

`docs/theorems/h-ver074-active-weyl-exponential-star-inverse.md`
`docs/theorems/h-ver075-binary-weyl-fourier-information-loss.md`
`docs/audits/h-aud017-active-weyl-odd-flip-source-fixedness.md`.

They own the two-axis exponential, binary information-loss, and odd-flip/fixedness clarification reproduced mathematically in Appendix B. They are provenance coverage, not upstream premises of the rank-ten reconstruction.

Comparison and full-ten firewalls. The comparison records have distinct roles:

- H-VER044: generic commuting-involution/Hadamard-square calculus;
- H-VER089: a distinct signed-root reciprocal $C_2 \times C_2$ packet;
- H-VER103: literal full-ten supported-corner firewall;
- H-VER104: active-label cyclicity and support firewall.

They prevent accidental identification of abstractly similar binary tables with the active-Weyl source object used here.

H-VER022. The downstream finite Fourier-to-sine information-loss direction used in Section 10 to type the GKS comparison is recorded in

`docs/theorems/h-ver022-dvj-brandas-d12rg-gks-fourier-n.md`.

It is not a source of the rank-ten fusion or Weyl reconstruction.

D.3 Exact executable receipts

The main reconstruction implementation is

`src/d12rg_hypergroup/a3_level2_generalized_chebyshev.py`

and its direct exhaustive regression is

`tests/test_h_ver068_a3_generalized_chebyshev_isomorphism.py`.

From the repository root it may be reproduced with

```
PYTHONPATH=src python3 -m pytest -q \
  tests/test_h_ver068_a3_generalized_chebyshev_isomorphism.py
```

The regression requires all 100 ordered fusion products and all 100 normalized probability-hypergroup products to agree exactly.

The surrounding finite claims have dedicated symbolic regressions:

`tests/test_h_ver064_zorb3_verlinde_hypergroup.py`
`tests/test_h_ver067_a3_level2_alcove.py`
`tests/test_h_ver070_a3_weyl_verlinde_fourier.py`
`tests/test_h_ver071_a3_simple_current_four_ality.py`
`tests/test_h_ver072_chebyshev_weyl_fourier_restriction.py`
`tests/test_h_ver073_a3_active_orbit_parity.py`
`tests/test_h_ver074_active_weyl_exponential.py`
`tests/test_h_ver075_binary_weyl_fourier.py`
`tests/test_h_ver082_chebyshev_halfstep_doublet.py`
`tests/test_h_aud017_active_weyl_odd_flip.py`.

They are regression certificates for exact finite identities, not numerical fits and not replacements for the mathematical statements in the paper.

Paper-specific structural checker. The machine-readable paper contract and source authority manifest are

`papers/weyl-verlinde-generalized-chebyshev/support/section0-dag-roots.json`
`papers/weyl-verlinde-generalized-chebyshev/support/source-manifest.json`,

and the paper-facing regression is

`tests/test_h_pap008_preprint_content.py`.

Section 0 retains both the visible theorem/audit owner ledger and the adjacent machine metadata. This appendix supplies the repository paths, historical notation translation, and executable receipts without replacing the front-facing Section-0 provenance contract.

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