the switch closes. Find the current through the voltage source during the interval $-\infty < t < \infty$, assuming the capacitor to have been initially uncharged.

18 At $t = -1$ capacitors C_1 and C_2 of Fig. 11.7 were seen to be discharging through the circuit containing the resistor R. At $t = 0$ the switch closed. Find the current through C_1 for all t.

19† Solve the following differential equations for $y(t)(t > 0)$, subject to initial values $y(0+), y'(0+), \ldots$.

a. $y'(t) + y(t) = 1$ [Ans: $y = 1 + [y(0+) - 1]e^{-t}$]

b. $y''(t) + 5y'(t) + 6y(t) = 6$ [Ans: $y = 1 + [3y(0+) + y'(0+) - 3]e^{-2t} + [2 - y'(0+) - 2y(0+)]e^{-3t}$]

c. $y''(t) + 2y'(t) + y(t) = 2\sin t$ [Ans: $y = [1 + y(0+) + y'(0+)]te^{-t} + [1 + y(0+)]e^{-t} - \cos t$

20 For each of the above problems derive an electric circuit obeying the same equation. Deduce driving functions for all t that would produce the responses quoted for $t > 0$.

21 Find four different time functions all having $[(p-2)(p-1)(p+1)]^{-1}$ as their Laplace transform. For each time function state the strip of convergence.

22 Show that the Laplace transform of the staircase function $H(t) + H(t - T) + H(t - 2T) + \cdots$ is given by

$$\frac{1}{p(1 - e^{-Tp})} \qquad 0 < \operatorname{Re} p.$$

23 A voltage $v(t)H(t)$ is applied to a circuit that already contained energy before $t = 0$. As no information about the stimulus has been suppressed, no compensating postinitial data are needed. However, in order to get the total behavior $f(t)$ it will be necessary to know about the natural behavior that is going on independently. It may be that the full history of earlier energy injection is unavailable, but it will suffice to know the situation immediately prior to the application of the voltage at $t = 0$. For example, the energy stored in each inductor and capacitor at $t = 0-$ might be given; but suppose here that a sufficient number of preinitial values of the behavior are given: $f(0-)$, $f'(0-)$, $f''(0-)$, $\ldots$. Show that the *total* behavior can be conveniently calculated by means of a special form of the Laplace transform defined by

$$F_-(p) = \int_{0-}^{\infty} f(t)e^{-pt}\,dt.$$

Work out the theorems for this transform, deducing, for example, that the derivative theorem is $f'(t) \supset pF_-(p) - f(0-)$.

† For a well-organized list of transforms of ratios of polynomials with denominators up to degree 5, which is the practical tool for solving this sort of problem once a sound knowledge of transform methods is attained, see P. A. McCollum and B. F. Brown: "Laplace Transform Tables and Theorems," Holt, Rinehart and Winston, New York, 1965.

Chapter 12 Relatives of the Fourier transform

Many of the linear transforms in common use have a direct connection with either the Fourier or the Laplace transform. The closest relationship is with the generalizations of the Fourier transform to two or more dimensions, and with the Hankel transforms of the zero and higher orders, into which the multidimensional Fourier transforms degenerate under circumstances of symmetry. The Mellin transform is illuminated by previous study of the Laplace transform and is the tool by which the fundamental theory of Fourier kernels is constructed. Particularly impressive is the simplification of the Hilbert transform when it is studied by the Fourier transform, and finally there is the intimate relationship whereby the Abel, Fourier, and Hankel transformations, applied in succession, regenerate the original function.

The two-dimensional Fourier transform

The variable x may stand for some physical quantity such as time or frequency, which is essentially one-dimensional, or it may be the coordinate in a one-dimensional physical system such as a stretched string or an electrical transmission line. However, in cases which are two dimensional—stretched membranes, antennas and arrays of antennas, lenses and diffraction gratings, pictures on television screens, and so on—more general formulas apply.

A two-dimensional function $f(x,y)$ has a two-dimensional transform $F(u,v)$, and between the two the following relations exist:

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i2\pi(ux+vy)}\,dx\,dy$$

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{i2\pi(ux+vy)}\,du\,dv.$$

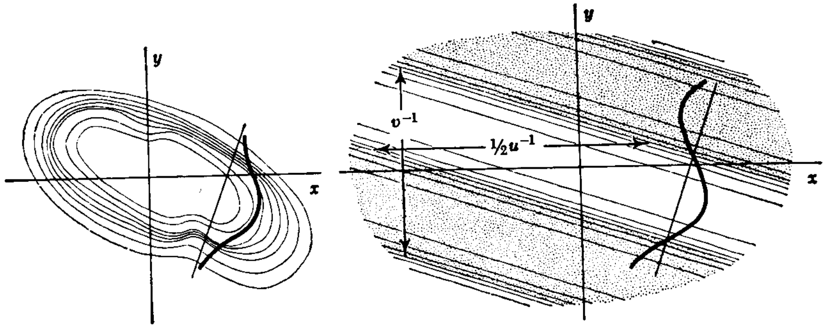


Fig. 12.1 A mountain (left) and a prominent Fourier component thereof (right).

These equations describe an analysis of the two-dimensional function $f(x,y)$ into components of the form $\exp [i2\pi(ux + vy)]$. Since any such component can be split into cosine and sine parts, we may begin by considering a cosine component $\cos [2\pi(ux + vy)]$.

As an example of a two-dimensional function consider the height of the ground at the geographical point (x,y), for example, over the area occupied by the mountain which is conventionally represented in Fig. 12.1 by contours of constant height. The function $\cos [2\pi(ux + vy)]$ represents a cosinusoidally corrugated land surface whose contours of constant height coincide with lines whose equation is

$$ux + vy = \text{const.}$$

The corrugations face in a direction that makes an angle $\arctan (v/u)$ with the x axis, and their wavelength is $(u^2 + v^2)^{-\frac{1}{2}}$. If a section is made through the corrugations, in the x direction, it will undulate with a frequency of u cycles per unit of x. Similarly, v may be interpreted as the number of cycles per unit of y, in the y direction.

In Fig. 12.1 a prominent Fourier component of the mountain is shown. In the transform domain the complex component is characterized in wavelength and orientation by the point (u,v) in the uv plane and its amplitude by $F(u,v)$. The interpretation of u and v as spatial frequencies is emphasized by dimensioning u^{-1} and v^{-1}, the wavelengths of sections taken in the x and y directions, respectively (see Fig. 12.1). The second of the Fourier relations quoted above asserts that a summation of corrugations of appropriate wavelengths and orientations, taken with suitable amplitudes, can reproduce the original mountain. The sinusoidal components, which must also be included, allow for the possibility that the corrugations may have to be slid into appropriate spatial phases.

Two-dimensional convolution

The convolution integral of two two-dimensional functions $f(x,y)$ and $g(x,y)$ is defined by

$$f ** g = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x',y')g(x - x', y - y')\, dx'\, dy'.$$

Thus one of the functions is rotated half a turn about the origin by reversing the sign of both x and y, displaced, and multiplied with the other function, and the product is then integrated to obtain the value of the convolution integral for that particular displacement.

The two-dimensional autocorrelation function is formed in the same way save that the sign reversal is omitted; thus

$$f \star\star\, g = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x',y')g(x + x', y + y')\, dx'\, dy'.$$

It is often convenient to be able to perceive ways in which a given function can be expressed as a convolution. For example, the two-dimensional function

$$^2\Pi(x,y) = \begin{cases} 1 & |x| \text{ and } |y| < \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$$

may be expressed as a product or as a convolution:

$$^2\Pi(x,y) = \Pi(x)\Pi(y) = [\Pi(x)\ \delta(y)] * [\Pi(y)\ \delta(x)].$$

These two possibilities are illustrated in Fig. 12.2.

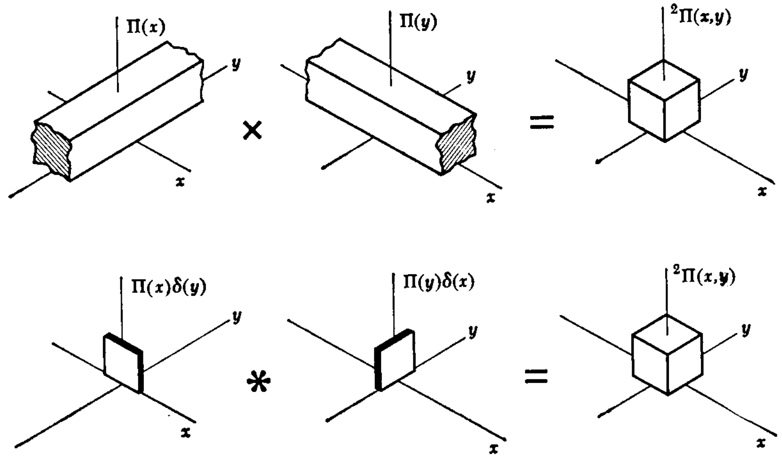


Fig. 1.22 Expressing a two diminsional function as a product and as a convolution.

The theorems pertaining to the one-dimensional transform generalize readily, as shown briefly in Table 12.1.

Figure 12.3 illustrates a number of two-dimensional Fourier transforms.

The Hankel transform

Two-dimensional systems may often show circular symmetry, for example, optical systems are often constructed from components that, in themselves, are circularly symmetrical. Then again, waves spreading out in two dimensions from a source of energy exhibit symmetry for natural reasons. It may be expected that in these cases a simplification will result, for one radial variable will suffice in place of the two independent variables x and y. The appropriate expression of such problems is in terms of the Hankel transform, a one-dimensional transform with Bessel function kernel.

When circular symmetry exists, that is, when

$$f(x,y) = \mathsf{f}(r),$$

where

$$r^2 = x^2 + y^2,$$

Table 12.1 Theorems for the two-dimensional Fourier transform

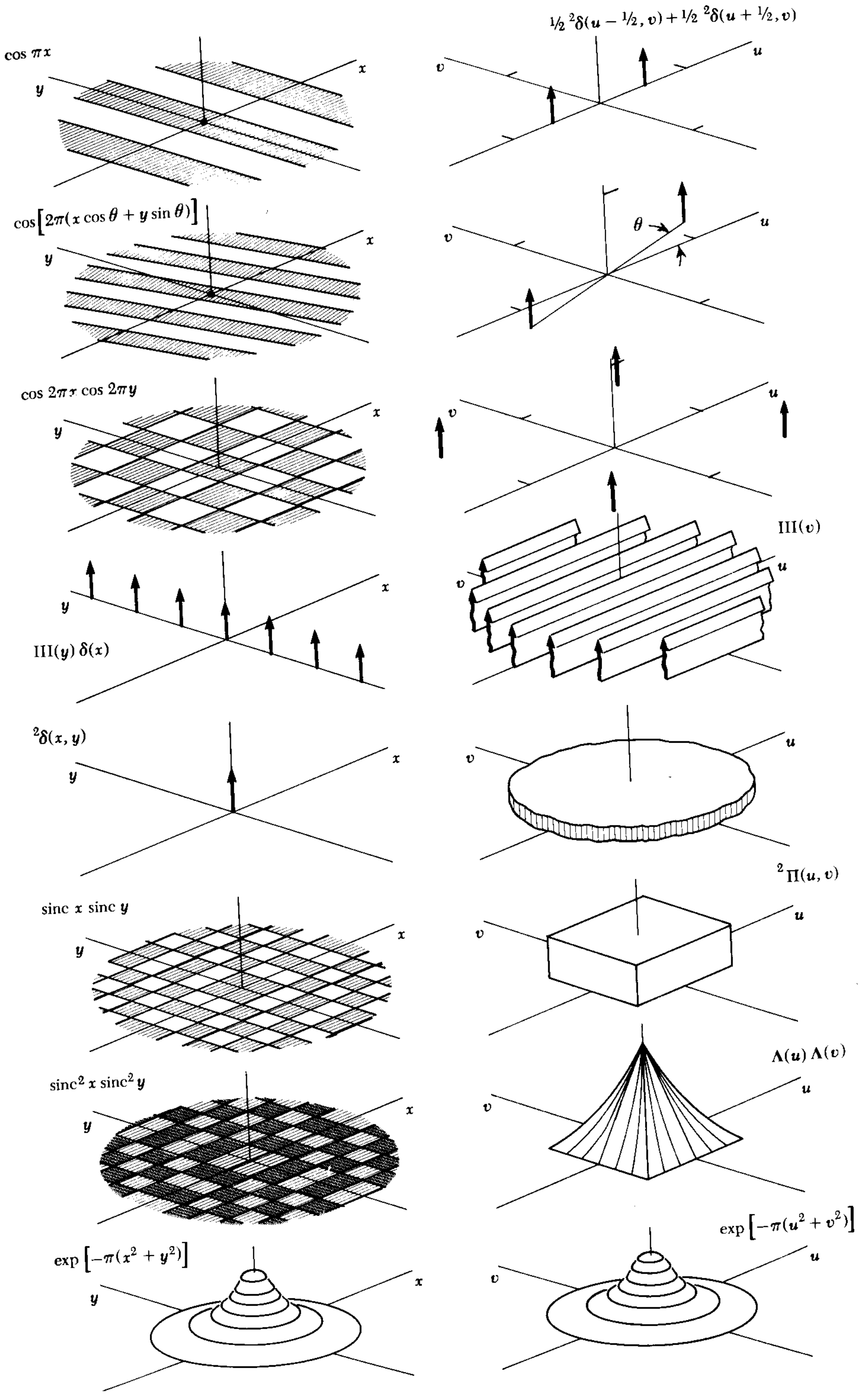
Theorem	$f(x,y)$	$F(u,v)$
Similarity	$f(ax,by)$	$\frac{1}{\lvert ab\rvert}F\left(\frac{u}{a},\frac{v}{b}\right)$
Addition	$f(x,y)+g(x,y)$	$F(u,v)+G(u,v)$
Shift	$f(x-a,\ y-b)$	$e^{-2\pi i(au+bv)}F(u,v)$
Modulation	$f(x,y)\cos\omega x$	$\frac{1}{2}F\left(u+\frac{\omega}{2\pi},v\right)+\frac{1}{2}F\left(u-\frac{\omega}{2\pi},v\right)$
Convolution	$f(x,y) * g(x,y)$	$F(u,v)G(u,v)$
Autocorrelation	$f(x,y) * f^*(-x,-y)$	$\lvert F(u,v)\rvert^2$
Rayleigh	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\lvert f(x,y)\rvert^2\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\lvert F(u,v)\rvert^2\,du\,dv$	
Power	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)g^*(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)G^*(u,v)\,du\,dv$	
Parseval	$\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\lvert f(x,y)\rvert^2 = \sum\sum a_{mn}^2$, where $F(u,v) = \sum\sum a_{mn}\,{}^2\delta(u-m,\ v-n)$	
Differentiation	$\left(\frac{\partial}{\partial x}\right)^m\left(\frac{\partial}{\partial y}\right)^n f(x,y)$	$(2\pi iu)^m(2\pi iv)^nF(u,v)$
	$\frac{\partial}{\partial x}f(x,y) = f'_x(x,y)$	$2\pi iuF(u,v)$
	$\frac{\partial}{\partial y}f(x,y) = f'_y(x,y)$	$2\pi ivF(u,v)$
	$\frac{\partial^2}{\partial x^2}f(x,y) = f''_{xx}(x,y)$	$-4\pi^2u^2F(u,v)$
	$\frac{\partial^2}{\partial y^2}f(x,y) = f''_{yy}(x,y)$	$-4\pi^2v^2F(u,v)$
	$\frac{\partial^2}{\partial x\,\partial y}f(x,y) = f''_{xy}(x,y)$	$-4\pi^2uvF(u,v)$
	$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)f(x,y)$	$-4\pi^2(u^2+v^2)F(u,v)$
Definite integral	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dx\,dy = F(0,0)$	
First moments	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xf(x,y)\,dx\,dy = \frac{1}{-2\pi i}F'_u(0,0)$	
	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x\cos\theta+y\sin\theta)f(x,y)\,dx\,dy = \frac{1}{-2\pi i}[\cos\theta\,F'_u(0,0)+\sin\theta\,F'_v(0,0)]$	
Center of gravity	$\langle x\rangle = \frac{F'_u(0,0)}{-2\pi iF(0,0)}$	$\langle y\rangle = \frac{F'_v(0,0)}{-2\pi iF(0,0)}$
Second moments	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x^2f(x,y)\,dx\,dy = \frac{F''_{uu}(0,0)}{-4\pi^2}$	
	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xyf(x,y)\,dx\,dy = \frac{F''_{uv}(0,0)}{-4\pi^2}$	
	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x^2+y^2)f(x,y)\,dx\,dy = -\frac{1}{4\pi^2}[F''_{uu}(0,0)+F''_{vv}(0,0)]$	
Equivalent width	$\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\,dx\,dy}{f(0,0)} = \frac{F(0,0)}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)\,du\,dv}$	
Finite differences†	$\Delta_x f(x,y)$	$i2\sin\pi u\,F(u,v)$
	$\Delta_{xy}{}^2f(x,y)$	$-4\sin\pi u\sin\pi v\,F(u,v)$
	$\Delta_{xx}{}^2f(x,y)$	$-4(\sin\pi u)^2F(u,v)$
Running means	$\left[\Pi\left(\frac{x}{a}\right)\Pi\left(\frac{y}{b}\right)\right] * f(x,y)$	$ab\,\mathrm{sinc}\,au\,\mathrm{sinc}\,bv\,F(u,v)$
Separable product	$f(x)g(y)$	$F(u)G(v)$

† The finite differences in the table are defined as follows:

$\Delta_x f(x,y) = f(x+\frac{1}{2},\ y) - f(x-\frac{1}{2},\ y)$

$\Delta_{xy}{}^2f(x,y) = f(x+\frac{1}{2},\ y+\frac{1}{2}) - f(x-\frac{1}{2},\ y+\frac{1}{2}) - f(x+\frac{1}{2},\ y-\frac{1}{2}) + f(x-\frac{1}{2},\ y-\frac{1}{2})$

$\Delta_{xx}{}^2f(x,y) = f(x+1,\ y) - 2f(x,y) + f(x-1,\ y).$



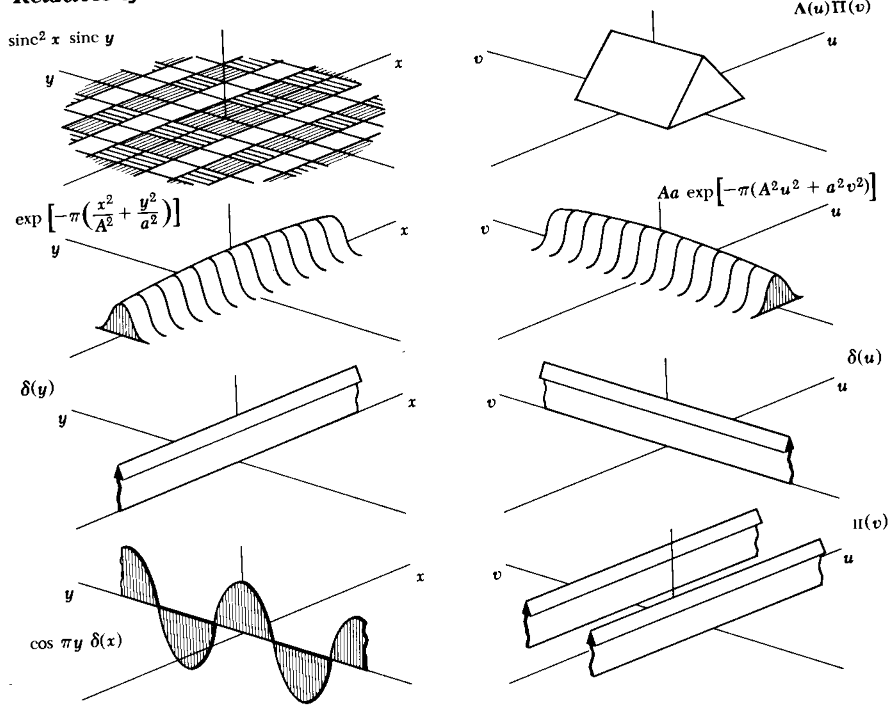


Fig. 12.3 Some two-dimensional Fourier transforms.

then $F(u,v)$ proves also to be circularly symmetrical; that is,

$$F(u,v) = \mathbf{F}(q),$$

where

$$q^2 = u^2 + v^2.$$

To show this, change the transform formula to polar coordinates and integrate over the angular variable.[7] Then the relations between the two one-dimensional functions $\mathbf{f}(r)$ and $\mathbf{F}(q)$ are

[7] That is,

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-i2\pi(xu+yv)}\,dx\,dy &= \int_0^{\infty}\int_0^{2\pi} \mathbf{f}(r)e^{-i2\pi qr\cos(\theta-\phi)}r\,dr\,d\theta \\ &= \int_0^{\infty} \mathbf{f}(r)\left[\int_0^{2\pi} e^{-i2\pi qr\cos\theta}\,d\theta\right] r\,dr \\ &= 2\pi\int_0^{\infty} \mathbf{f}(r)J_0(2\pi qr)r\,dr\end{aligned}$$

where $x + iy = re^{i\theta}$, $u + iv = qe^{i\phi}$ and we have used the relation

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{-iz\cos\beta}\,d\beta.$$

$$\mathbf{F}(q) = 2\pi \int_0^\infty \mathbf{f}(r) J_0(2\pi qr) r \, dr$$

and

$$\mathbf{f}(r) = 2\pi \int_0^\infty \mathbf{F}(q) J_0(2\pi qr) q \, dq.$$

We refer to $\mathbf{F}(q)$ as the Hankel transform (of zero order) of $\mathbf{f}(r)$ and note that the transformation is strictly reciprocal, as was the case when the kernels were cos and sin. The kernel J_0, together with cos, sin, and others, is referred to as a Fourier kernel in the broad sense of a kernel associated with a reciprocal transform.

The factors 2π in the above formulas may be canceled by suitable redefinition of the variables, but their retention follows logically from the form adopted for Fourier transforms. In physical situations the 2π in parentheses will be found to result from the measurement of q in whole cycles per unit of r. The 2π before the integral sign comes from the element of area $2\pi r \, dr$.

A number of zero-order Hankel transforms are shown as two-dimensional Fourier transforms in Fig. 12.4. Table 12.2. lists various Hankel transforms for reference.

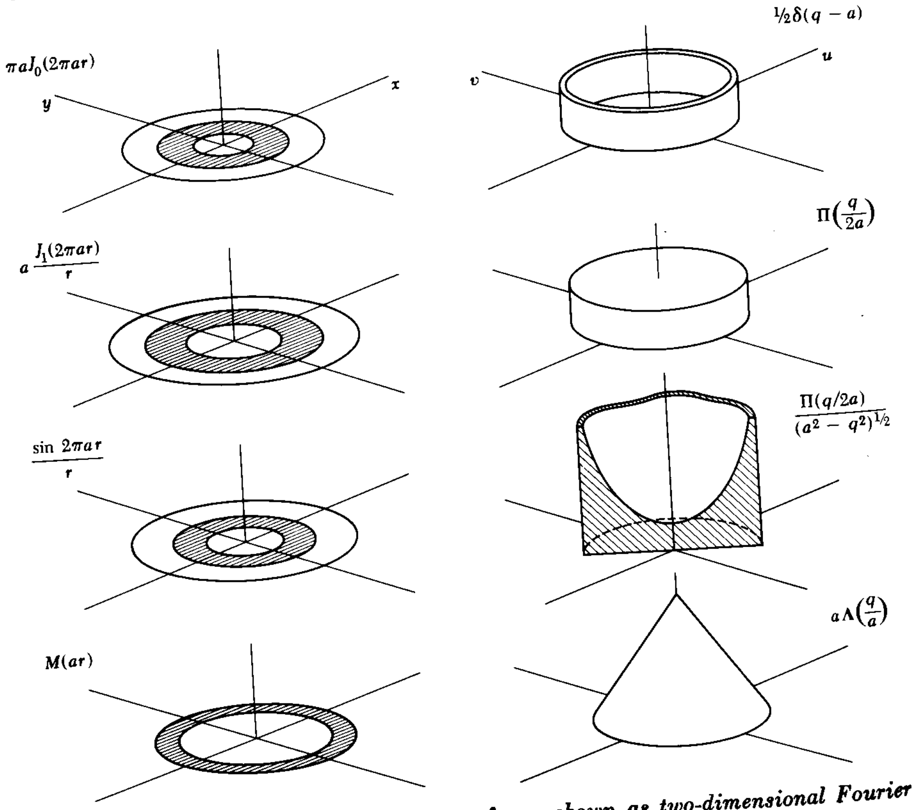


Fig. 12.4 Some zero-order Hankel transforms shown as two-dimensional Fourier transforms.

Table 12.2 Some Hankel transforms

$\mathbf{f}(r)$	$\mathbf{F}(q)$
$\Pi\left(\frac{r}{2a}\right)$	$\frac{aJ_1(2\pi aq)}{q}$
$\frac{\sin 2\pi ar}{r}$	$\frac{\Pi(q/2a)}{(a^2-q^2)^{\frac{1}{2}}}$
$\frac{1}{2}\delta(r-a)$	$\pi a J_0(2\pi aq)$
$M(ar)$†	$a^{-2}\Lambda\left(\frac{q}{a}\right)$
$e^{-\pi r^2}$	$e^{-\pi q^2}$
$\frac{1}{(a^2+r^2)^{\frac{1}{2}}}$	$\frac{e^{-2\pi aq}}{q}$
$\frac{1}{(a^2+r^2)^{\frac{3}{2}}}$	$\frac{2\pi e^{-2\pi aq}}{a}$
$\frac{1}{a^2+r^2}$	$2\pi K_0(2\pi aq)$
$\frac{2a^2}{(a^2+r^2)^2}$	$4\pi^2 aqK_1(2\pi aq)$
$\frac{4a^4}{(a^2+r^2)^3}$	$4\pi^2 aqK_1(2\pi aq) + 4\pi^3 a^2q^2K_0(2\pi aq)$
$(a^2-r^2)\,\Pi\left(\frac{r}{2a}\right)$	$\frac{a^2J_2(2\pi aq)}{\pi q^2}$
$\frac{1}{r}$	$\frac{1}{q}$
e^{-ar}	$\frac{2\pi a}{(4\pi^2q^2+a^2)^{\frac{3}{2}}}$
$\frac{e^{-ar}}{r}$	$\frac{2\pi}{(4\pi^2q^2+a^2)^{\frac{1}{2}}}$
$\frac{\delta(r)}{\pi r} = {}^2\delta(x,y)$	1
$\frac{\Pi\left(\frac{r}{2a}\right)^{**2}}{2a^2} = \left[\cos^{-1}\frac{r}{2a} - \frac{r}{2a}\left(1-\frac{r^2}{4a^2}\right)^{\frac{1}{2}}\right]\Pi\left(\frac{r}{4a}\right)$	$\frac{[J_1(2\pi aq)]^2}{2q^2}$
$-4\pi^2r^2f(r)$	$\left(\frac{d^2F}{dq^2}+\frac{1}{q}\frac{dF}{dq}\right) = \nabla^2F$
$r^2e^{-\pi r^2}$	$\left(\frac{1}{\pi}-q^2\right)e^{-\pi q^2}$

† In this table $M(x) = 2\pi\left[x^{-3}\int_0^x J_0(x)\,dx - x^{-2}J_0(x)\right]$.

Table 12.3 Theorems for the Hankel transform

Theorem	$\mathsf{f}(r)$	$\mathsf{F}(q)$
Similarity	$\mathsf{f}(ar)$	$a^{-2}\mathsf{F}\left(\frac{q}{a}\right)$
Addition	$\mathsf{f}(r) + \mathsf{g}(r)$	$\mathsf{F}(q) + \mathsf{G}(q)$
Shift	Shift of origin destroys circular symmetry	
Convolution	$\int_0^\infty \int_0^{2\pi} \mathsf{f}(r')\mathsf{g}(R)r'\,dr'\,d\theta$ $(R^2 = r^2 + r'^2 - 2rr'\cos\theta)$	$\mathsf{F}(q)\mathsf{G}(q)$
Rayleigh	$\int_0^\infty \lvert\mathsf{f}(r)\rvert^2 r\,dr = \int_0^\infty \lvert\mathsf{F}(q)\rvert^2 q\,dq$	
Power	$\int_0^\infty \mathsf{f}(r)\mathsf{g}^*(r)r\,dr = \int_0^\infty \mathsf{F}(q)\mathsf{G}^*(q)q\,dq$	
Differentiation	Exercise for student	
Definite integral	$2\pi\int_0^\infty \mathsf{f}(r)r\,dr = \mathsf{F}(0)$	
Second moment	$2\pi\int_0^\infty r^2\mathsf{f}(r)r\,dr = \dfrac{\mathsf{F}''(0)}{-2\pi^2}$	
Equivalent width	$\dfrac{2\pi\int_0^\infty \mathsf{f}(r)r\,dr}{\mathsf{f}(0)} = \dfrac{\mathsf{F}(0)}{2\pi\int_0^\infty \mathsf{F}(q)q\,dq}$	

Many of the theorems for the two-dimensional Fourier transform can be restated in terms of the Hankel transform. The names of the corresponding Fourier theorems are listed in Table 12.3 to allow comparison.

Fourier kernels

Let two functions f and g be related through the following integral equation whose kernel is k:

$$g(s) = \int_0^\infty f(x)k(s,x)\,dx,$$

and let the kernel be such that a reciprocal relationship also holds; that is,

$$f(x) = \int_0^\infty g(s)k(s,x)\,ds.$$

We know that $2\cos 2\pi ax$ and $2\sin 2\pi ax$ are such kernels, and a whole further set is furnished by a theorem established by Hankel, namely,[2]

$$g(x) = \int_0^\infty ds\,(xs)^{\frac{1}{2}}J_\nu(xs)\int_0^\infty g(x)(xs)^{\frac{1}{2}}J_\nu(xs)\,dx,$$

whence $$G(s) = \int_0^\infty g(x)(xs)^{\frac{1}{2}}J_\nu(xs)\,dx,$$

[2] Where $f(x)$ is discontinuous, the left-hand side should be replaced by $\frac{1}{2}[f(x+0) + f(x-0)]$.

and conversely, $g(x) = \int_0^\infty G(s)(xs)^{\frac{1}{2}}J_\nu(xs)\,ds.$

By splitting off a factor $s^{\frac{1}{2}}$ from $G(s)$ and $x^{\frac{1}{2}}$ from $g(x)$, that is, by putting $G(s) = s^{\frac{1}{2}}F(s)$ and $g(x) = x^{\frac{1}{2}}f(x)$, we obtain the following alternative expressions of the above formulas:

$$F(s) = \int_0^\infty xf(x)J_\nu(xs)\,dx$$
$$f(x) = \int_0^\infty sF(s)J_\nu(xs)\,ds.$$

The case in which $\nu = 0$ was derived earlier from the two-dimensional Fourier transform under conditions of circular symmetry.

It is interesting that by taking $\nu = \pm\frac{1}{2}$ and using the relations

$$J_{\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\sin z, \qquad J_{-\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\cos z,$$

we recover the known kernels $2\cos 2\pi ax$ and $2\sin 2\pi ax$, which shows that the cosine and sine transform formulas are included in Hankel's theorem.

The three-dimensional Fourier transform

Undoubtedly physical systems have three dimensions, but for reasons of theoretical tractability, one seeks simplifications. The classical example in which Fourier analysis in three dimensions has nevertheless had to be faced is the diffraction of X rays by crystals. The formulas are

$$F(u,v,w) = \int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty f(x,y,z)e^{-i2\pi(xu+yv+zw)}\,dx\,dy\,dz$$
$$f(x,y,z) = \int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty F(u,v,w)e^{i2\pi(ux+vy+wz)}\,du\,dv\,dw.$$

Multidimensional transforms, should they be encountered, will be recognized without difficulty. By taking $\mathbf{x}$ and $\mathbf{s}$ to be vectors whose components are $(x_1,x_2, \ldots)$ and $(s_1,s_2, \ldots)$, we have the following convenient vector notation for n-dimensional transforms:

$$F(\mathbf{s}) = \iint \cdots \int_{-\infty}^\infty f(\mathbf{x})e^{-i2\pi\mathbf{x}\cdot\mathbf{s}}\,dx_1\,dx_2 \ldots dx_n.$$

In cylindrical coordinates r, θ, z where

$$x + iy = re^{i\theta},$$

the three-dimensional transform may be expressed in terms of the transform variables s, ϕ, w, where

$$u + iv = se^{i\phi},$$

by the formulas

$$G(s,\phi,w) = \int_0^\infty \int_0^{2\pi} \int_{-\infty}^\infty g(r,\theta,z) e^{-i2\pi[sr\cos(\theta-\phi)+wz]} r\, dr\, d\theta\, dz$$

$$g(r,\theta,z) = \int_0^\infty \int_0^{2\pi} \int_{-\infty}^\infty G(s,\phi,w) e^{i2\pi[sr\cos(\theta-\phi)+wz]} s\, ds\, d\phi\, dw.$$

These results are derivable directly from the basic formulas by substituting

$$g(r,\theta,z) = f(x,y,z)$$

and

$$G(s,\phi,w) = F(u,v,w).$$

Under circular symmetry, that is, when f is independent of θ (and hence F independent of ϕ), we find by writing

$$h(r,z) = f(x,y,z)$$

and

$$H(s,w) = F(u,v,w)$$

that

$$H(s,w) = 2\pi \int_0^\infty \int_{-\infty}^\infty h(r,z) J_0(2\pi sr) e^{-i2\pi wz} r\, dr\, dz$$

$$h(r,z) = 2\pi \int_0^\infty \int_{-\infty}^\infty H(s,w) J_0(2\pi sr) e^{i2\pi wz} s\, ds\, dw.$$

To obtain this result we use the formula derived earlier for the Hankel transform of zero order.

Under cylindrical symmetry, that is, when f is independent of both θ and z, being a function of r only, say $f(x,y,z) = \mathbf{k}(r)$ and $F(u,v,w) = K(s,w)$, then

$$K(s,w) = \mathbf{K}(s)\, \delta(w),$$

where

$$\mathbf{K}(s) = 2\pi \int_0^\infty \mathbf{k}(r) J_0(2\pi sr) r\, dr.$$

In spherical coordinates r,θ,ϕ, with transform variables s,Θ,Φ, we have

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$
$$u = s\sin\Theta\cos\Phi \qquad v = s\sin\Theta\sin\Phi \qquad w = s\cos\Theta.$$

Writing $f(x,y,z) = g(r,\theta,\phi)$ and $F(u,v,w) = G(s,\Theta,\Phi)$, we find

$$G(s,\Theta,\Phi) = \int_0^\infty \int_0^\pi \int_0^{2\pi} g(r,\theta,\phi)\, e^{-i2\pi sr[\cos\Theta\cos\theta + \sin\Theta\sin\theta\cos(\phi-\Phi)]} r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

$$g(r,\theta,\phi) = \int_0^\infty \int_0^\pi \int_0^{2\pi} G(s,\Theta,\Phi)\, e^{i2\pi sr[\cos\Theta\cos\theta + \sin\Theta\sin\theta\cos(\phi-\Phi)]} s^2 \sin\Theta\, ds\, d\Theta\, d\Phi.$$

With circular symmetry, that is, when $f(x,y,z)$ is independent of ϕ, we have, writing $f(x,y,z) = h(r,\theta)$ and $F(u,v,w) = H(s,\Theta)$,

$$H(s,\Theta) = 2\pi \int_0^\infty \int_0^\pi h(r,\theta) J_0(2\pi sr\sin\Theta\sin\theta) e^{-i2\pi sr\cos\Theta\cos\theta} r^2 \sin\theta\, dr\, d\theta.$$

With spherical symmetry, we have, writing $f(x,y,z) = k(r)$ and

$$F(u,v,w) = K(s),$$
$$K(s) = 4\pi \int_0^\infty k(r)\,\operatorname{sinc}(2sr)\, r^2\, dr,$$
$$k(r) = 4\pi \int_0^\infty K(s)\,\operatorname{sinc}(2sr)\, s^2\, ds.$$

A few examples of three-dimensional Fourier transforms are given in Table 12.4. Many more can be generated by noting that

$$f(x)g(y)h(z) \supset F(u)G(v)H(w),$$

where $f(x), F(u)$, and the like are one-dimensional Fourier transform pairs. This result is proved by expressing $f(x)g(y)h(z)$ in the form

$$f(x)\,\delta(y)\,\delta(z) * \delta(x)g(y)\,\delta(z) * \delta(x)\,\delta(y)h(z)$$

and then applying the convolution theorem in three dimensions. As various cases of this kind we have

$$\begin{aligned} f(x)g(y)h(z) &\supset F(u)G(v)H(w) \\ f(x)g(y) &\supset F(u)G(v)\,\delta(w) \\ f(x) &\supset F(u)\,\delta(v)\,\delta(w) \\ \mathbf{k}(r)h(z) &\supset \mathbf{K}(s)H(w) \\ \mathbf{k}(r) &\supset \mathbf{K}(s)\,\delta(w) \end{aligned}$$

Table 12.4 Some three-dimensional Fourier transforms†

$f(x,y,z)$		$F(u,v,w)$
${}^3\delta(x-a,\ y-b,\ z-c)$	point	$e^{i2\pi(au+bv+cw)}$
$e^{-\pi(x^2/a^2+y^2/b^2+z^2/c^2)}$	Gaussian	$abce^{-\pi(a^2u^2+b^2v^2+c^2w^2)}$
${}^3\Pi(x,y,z)$	cube	$\operatorname{sinc} u \operatorname{sinc} v \operatorname{sinc} w$
${}^2\Pi(x,y)$	bar	$\operatorname{sinc} u \operatorname{sinc} v\, \delta(w)$
$\Pi(x)$	slab	$\operatorname{sinc} u\, \delta(v)\, \delta(w)$
$\Pi(x)\Pi[(y^2+z^2)^{\frac{1}{2}}]$	disk	$\operatorname{sinc} u \dfrac{J_1[\pi(v^2+w^2)^{\frac{1}{2}}]}{2(v^2+w^2)^{\frac{1}{2}}}$
$\Pi\left(\dfrac{r}{2}\right)$	ball	$\dfrac{\sin 2\pi s - 2\pi s\cos 2\pi s}{2\pi^2 s^3}$
$(1-\|r\|)\Pi\left(\dfrac{r}{2}\right)$		$\dfrac{\pi}{3}\dfrac{12}{(2\pi)^4}\dfrac{2(1-\cos 2\pi s) - 2\pi s\sin 2\pi s}{s^4}$
$(1-r^2)\Pi\left(\dfrac{r}{2}\right)$		$\dfrac{8\pi}{(2\pi)^5}\dfrac{[3-(2\pi s)^2]\sin 2\pi s - 3(2\pi s)\cos 2\pi s}{s^5}$
$\dfrac{e^{-r/R}}{\frac{4}{3}\pi R^3}$		$\dfrac{6}{(1+4\pi^2R^2s^2)^2}$
$e^{-\pi r^2}$		$e^{-\pi s^2}$

† In this table $r^2 = x^2 + y^2 + z^2$ and $s^2 = u^2 + v^2 + w^2$.