Group Theory and the SO(3,1) Lorentz Group

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December 4, 2009

Abstract

In this paper, fundamental mathematical concepts in group theory are presented to form a foundation on which to study more specific groups. After providing a brief analysis of rotation groups, Lie groups, and group representations, these concepts are applied to the Lorentz group in physics. The isomorphism SO(3,1) $\simeq$ [SU(2) $\otimes$ SU(2)]/$\mathbb{Z}_2$ is shown explicitly and interpreted in terms of the transformation of particles identified by their spin quantum number. Further representations of the Lorentz group are then discussed with the implications to particle physics.

1 Introduction

Group theory as a mathematical study alone is a vast field that can provide insights into symmetries of systems working from a few basic principles that define what a group is and how one can work with it. Many mathematical concepts related to group theory were first developed by Gauss, Lagrange, Poincaré, and Klein in the context of number theory, algebra, and geometry. However, Evariste Galois was the first to endow a group with the meaning it holds in the current study [2]. However, it is usually credited to Eugene Wigner to have made group theory a useful tool in physics with his 1939 publication concerning the inhomogeneous Lorentz group [3]. From this point on, group theory has been developed to be an essential tool in studying relativistic quantum mechanics and symmetries of systems that may otherwise be inaccessible by other means. In this study, group theory is approached from the point of view of understanding the Lorentz group and certain properties it has that give insights into studying classes of fields and particles.

2 Group Theory Concepts

The mathematical study of groups is extensive but here, I will cover a few of the key concepts that are necessary to understanding the Lorentz group and the certain aspects of it that will be studied.

2.1 Group Definition

The fundamental concepts of defining a group are simple and can be concisely stated in four rules. A collection of elements G and an arbitrarily defined operation between two of these elements $(\cdot)$ if together they satisfy the conditions of [3]:

1. Closure : Given two elements $x, y \in G$,then the element $x \cdot y \in G$
2. Associativity : Given $x, y, z \in G$ then the order in which $\cdot$ is performed will not affect the outcome. Mathematically this is stated as
   $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
3. Identity element : There must exist some element $e \in G$ that when combined with an element $x \in G$ it obeys $x \cdot e = e \cdot x = x$
4. Inverse : There must exist an element $x^{-}-1 \in G$ for every $x \in G$ such that $x \cdot x^{-1} = x^{-1} \cdot x = x$

One simple example of this is using integer numbers (...-2,-1,0,1,2...) as the set of group elements with addition as the group operation. It is clear that when two elements of the set are combined by addition, the result will still be an integer. Further, consider associativity. If we have three integers (a, b, c) then it is clear that $(a+b)+c = a+(b+c)$. We find the integer 0 as the additive identity since $a + 0 = 0 + a = a$. Lastly, for any integer a, the inverse integer element is found to be $-a$ so that $a+(-a) = (-a)+a = 0$ to produce the identity of 0. Having shown that the four rules hold for integers, it is proven that integers with the operation of addition is a group.

2.2 Representation of Groups

However, to address more interesting questions, the groups become more complicated. In mathematics, the calculations can be performed using the intricate structure of group theory that is not developed greatly here. However, in physics when it becomes necessary to perform an actual calculation to obtain a measurable result, it is more convenient to work with groups as more tangible objects. This is known as the representation theory of groups, and in this context, the elements of an abstract group are treated as invertible linear transformations on a vector space (i.e. matrices) and the group operation is defined as the normal matrix multiplication as in linear algebra. One immediate advantage of using matrices to represent group elements is that the identity element can immediately be recognized as the unit matrix $\mathbb{1}$ and the inverse element of a group element M is simply its matrix inverse M^{-1}. Further, due to the associativity of matrix multiplication, the rule for associativity of group elements is satisfied. To ensure that a given set of elements is a group, the last consideration is to ensure closure of the multiplication of matrices within the group. As an example of closure once a group has been expressed in terms of matrices, consider the group of two dimensional vector rotations that preserves the magnitude of the vector. We can write the counterclockwise rotation of the vector $\vec{v_1}$ to $\vec{v_2}$ through an angle of θ as

$$\vec{v_2} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \vec{v_1} \tag{1}$$

and when two rotations (of θ and then ϕ operate successively, we can multiply them together to obtain

$$\begin{pmatrix} cos(\theta+\phi) & -sin(\theta+\phi) \\ sin(\theta+\phi) & cos(\theta+\phi) \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} cos\phi & -sin\phi \\ sin\phi & cos\phi \end{pmatrix} \tag{2}$$

This is simply a rotation through the angle $\theta + \phi$ which can be performed as a single rotation. Thus, this set of rotations, in the representation of matrices, is shown to be closed under matrix multiplication and is thus a group.

We denote this group as SO(2) for special (S) orthogonal (O) of dimension 2 in which special means that the for any $M \in SO(2)$, $det(M) = 1$ and orthogonal means that $MM^T = M^TM = \mathbb{1}$. This is a described as a subgroup of the larger group O(2) because the elements of SO(2) are a closed subset of those in O(2). This notion can be extended to n dimensions in which the group O(n) contains elements M that are either improper rotations ($det(M) = -1$) or proper rotations ($det(M) = 1$) where the latter comprises the elements of SO(n). Physically, this is the property of the Lorentz group elements, which are realized as proper rotations in the Minkowski vector space and so are classified as a physical subgroup of the larger Poincaré group.

2.3 Lie Groups

A further point of interest that must be addressed before examining the Lorentz group is that of Lie groups. This is because Lie groups are groups of continuous transformations that can be decomposed into a series of infinitesimal transformations and such is the case of the Lorentz group transformations.

One key point that distinguishes Lie groups from all continuous parameter groups is that it contains only the elements that lie close to the identity element of the group. Furthermore, it can be shown by considering the differential calculus of the continuous group that there exist a fundamental set of group elements called generators which can be combined with each other and themselves to produce any element in the group [8]. Furthermore, these generators G_α can be shown to obey a fundamental commutation relation $[G_\alpha, G_\beta] = iC_{\alpha\beta\gamma}G_\gamma$ in which the $C_{\alpha\beta\gamma}$ are called structure constants mathematically and depend on the context in which the Lie algebra is applied [3]. Thus, if such a commutation relation can be shown for the elements of the group, then closure of the group is satisfied due to the fact that it is a Lie group. Using these two ideas of proximity to the identity and generators of the group, then we can consider an infinitesimal transformation of this group as being of the form $A = \mathbb{1} - iG_\alpha\epsilon_\alpha$ where the infinitesimal parameter ϵ causes for a small transformation in the G_α direction. This can be extended to a continuous transformation over a finite value t by considering t to be broken into N individual infinitesimal transformations and compounding these together as in 3

$$A_{finite} = [\mathbb{1} - \frac{iG_\alpha t}{N}]^N = ([\mathbb{1} - \frac{iG_\alpha t}{N}]^{\frac{N}{-iGt}})^{iGt} \tag{3}$$

and by recognizing that in the limit that $N \Rightarrow \infty$ the expression inside the smooth parentheses is simply e, we find

$$A_{finite} = e^{-iGt} \tag{4}$$

One can visualize such a continuous transformation as motion through the a manifold defined by the parameters of the transformation, in the previous case t. As a step towards the Lorentz group, the rotation group SO(3) can be visualized as a sphere of radius π and three dimensional rotations trace out paths on this manifold [3].

3 The Lorentz Group

Although group theory is very powerful in many areas of physics where symmetries occur, one area where it plays a very prominent role is in describing transformations related to special relativity. This concerns the study of the Lorentz group, denoted by SO(3,1), and can be used to gain great insights into particle physics.

3.1 The Fundamental Lorentz Group

In a sense, it is an extension of the three dimensional SO(3) but to include a group of transformations involving time. These transformation are those of boosts that bring an observer from one inertial reference frame to another relativistically. The Lorentz group that is most commonly referred to is that concerning physical transformations in special relativity. However, this is really a subgroup of a larger group called the Poincaré group. In general, the Poincaré group elements obey the fundamental relation

$$g_{\mu\nu} = \Lambda_{\rho\mu}\Lambda_{\sigma\nu}g_{\rho\sigma} \tag{5}$$

which can be imagined at the metric $g_{\rho\sigma}$ being invariant under Lorentz transformations from one inertial frame to another. By considering the determinant of this equation and the case of this equation for $\mu = \nu = 0$, respectively, the following conditions can be found.

$$det(\Lambda_{\rho\mu}) = \pm 1 \quad \text{and} \quad \Lambda_{00} \geq 1 \text{ or } \Lambda_{00} \leq -1 \tag{6}$$

These conditions divide the Poincaré group into four subsets by the four choices of the combinations of these two conditions. The group that will be studied is that in which the determinant is 1 (proper) and $\Lambda_{00} \geq 1$ (orthochronous). In fact, this is the only subset of the Poincaré group which is a subgroup as well because only it contains the identity element $\mathbb{1}$ [1]. This is the group to which many refer as SO(3,1) Lorentz group and is what will be studied from here. The group elements $\Lambda_{\rho\mu}$ will be represented by 4×4 matrices. By examining 6 which is symmetric under interchange of μ and ν, the sixteen elements of the transformation are constrained such that any transformation can be described by six parameters. These six parameters can be thought of in terms of three corresponding to rotations about three orthogonal directions in space and boosts along these directions.

Because this Lorentz group is describing the set of continuous physical transformation of boosts and rotations, it is classified as a Lie group and so has an infinitesimal transformation representation of $D = 1 + \frac{1}{2}\epsilon_{\mu\nu}M_{\mu\nu}$ in which $M_{\mu\nu}$ are the generators defined as the antisymmetric differential operator $M_{\mu\nu} = \partial_\mu x_\nu - \partial_\nu x_\mu$. Using this definition, it can be shown that these generators commute as

$$[M^{\mu\nu}, M^{\sigma\rho}] = i(g^{\nu\sigma}M^{\mu\rho} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + g^{\mu\rho}M^{\nu\sigma}) \tag{7}$$

and by identifying the rotation and boost operations in terms of the generators respectively as

$$L_i = -\frac{1}{2}\epsilon_{ijk}M_{jk} \tag{8}$$

$$K_i = M_{0i} \tag{9}$$

the commutation relations for the Lorentz group can be obtained using 7 as

$$[L_i, L_j] = i\epsilon_{ijk}L_k \tag{10}$$

$$[K_i, L_j] = i\epsilon_{ijk}K_k \tag{11}$$

$$[K_i, K_j] = -\epsilon_{ijk}L_k \tag{12}$$

Examining these commutation relations, closure is realized because each fulfills the form of the general Lie group commutation relation. In addition, the first is the same as for normal angular momentum generators. However, it is interesting to note that the commutation of two boosts results in a rotation. This is not at first obvious. However, if one considers a four dimensional space in with three spatial directions and one of time, then a boost can be considered as a rotation in imaginary time and so the commutator indeed returns another rotation in this same space, but now on the spatial directions. This further shows that boosts alone do not form a subgroup because they do not fulfill the commutation relation previously found for Lie algebras and so are not closed by themselves.

The next step is to find the form of finite Lorentz transformations. By considering the infinitesimal transformation D upon the identification of specific generators with boosts and rotations. Then consider the infinitesimal ϵ elements to be unique for the different boosts ($\epsilon^i = n^i \delta y$) and rotations ($\epsilon^i = m^i \delta\theta$) in which δy is a convenient parameter called rapidity used to give the amount by which the boost occurs and $\vec{m}$ and $\vec{n}$ give the unit directions of the respective transformations. By making this association, the exponential map from the group generators of infinitesimal transformations to finite transformations as in 13 [1].

$$D_{finite} = e^{\vec{K}\cdot\vec{n}y + i\vec{L}\cdot\vec{m}\theta} \tag{13}$$

This is the general operator that would act on a physical object to produce a Lorentz transformation. As before, this finite transformation, in the sense of a Lie group, can be visualized as tracing out a path on the associated manifold. In this case, the manifold is similar to the sphere in the case of SO(3) but because there is the additional dimension of time, one must visualize the paths being traced out on a higher dimensional hypersphere.

3.2 Alternative Representations

Having found a suitable representation for SO(3,1) boosts and rotations in terms of the infinitesimal generators $M_{\mu\nu}$, it is interesting to search for alternative representations of the Lorentz group. This can be done by simply making a change of bases by defining the new operators $J_i^{\pm} = \frac{1}{2}(L_i \pm iK_i)$. The commutation relations between these are calculated explicitly as

$$\begin{aligned}
[J_i^+, J_i^-] &= \frac{1}{4}([L_i, L_j] - i[L_i, K_j] + i[K_i, L_j] - [K_i, K_j]) \\
&= \frac{1}{4}(i\epsilon_{ijk}L_k - \epsilon_{ijk}K_k - \epsilon_{jik}K_k - i\epsilon_{ijk}L_k) \\
&= 0
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
[J_i^{\pm}, J_i^{\pm}] &= \frac{1}{4}([L_i, L_j] \pm i[L_i, K_j] \pm i[K_i, L_j] - [K_i, K_j]) \\
&= \frac{1}{4}(i\epsilon_{ijk}L_k \pm \epsilon_{jik}K_k \mp \epsilon_{ijk}K_k + i\epsilon_{ijk}L_k) \\
&= \frac{1}{4}(2i\epsilon_{ijk}L_k \mp 2\epsilon_{ijk}K_k) \\
&= \frac{1}{2}i\epsilon_{ijk}(L_k \pm iK_k) \\
&= \frac{1}{2}i\epsilon_{ijk}J_k^{\pm}
\end{aligned} \tag{15}$$

From these calculations, it is clear that the two groups (J^+ and J^-) do not overlap due to the fact that their commutator vanishes. Furthermore, each of these group forms its own group which obeys the same Lie group commutation rule as for the rotation group SO(3). And because SO(3) is doubly covered by the group SU(2), by removing half of the elements as SU(2)/$\mathbb{Z}_2$, we can consider this group to be isomorphic to SO(3), meaning that there is a one to one map between the elements in each group. Therefore, we have shown that the following relation holds.

$$SO(3,1) \simeq [SU(2) \otimes SU(2))]/\mathbb{Z}_2 \tag{16}$$

in which each of the SU(2) groups correspond to one of the J^+ and J^- group elements [5].

This is a fundamental isomorphism in the Lorentz group that can be used to understand implications in particle physics. By considering the new basis of $J^{\pm}$, we can rewrite the old K, L group elements as

$$L_i = J_i^+ + J_i^- \tag{17}$$

$$iK_i = J_i^+ - J_i^- \tag{18}$$

and substituting these in to the finite Lorentz transformation operator 13 we find

$$\begin{aligned} D_{finite} &= e^{-(\vec{J^+} - \vec{J^-})\cdot\vec{n}y + i(\vec{J^+} + \vec{J^-})\cdot\vec{m}\theta} \\ &= e^{(\vec{n}y + i\vec{m}\theta)\cdot\vec{J^+}} e^{(-\vec{n}y + i\vec{m}\theta)\cdot\vec{J^-}} \end{aligned} \tag{19}$$

By representing the transformation in this form, it can be seen that there are two distinct pieces of a transformation, each corresponding to one of the SU(2) groups found previously. By considering these as different representations of the Lorentz group, we have decomposed it into what are called irreducible representations. Applying this to physics, the manner in which these representations are labeled is labeled by the set of values (j^+, j^-) which correspond to the angular momentum of the physical object associated with that representation. Further, the given (j^+, j^-) representation will contain objects with $(2j^+{+}1)(2j^-{+}1)$ components [1].

Two such related representations are the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations which correspond to the individual transformations of

$$D_{(\frac{1}{2},0)} = e^{(-\vec{n}y + i\vec{m}\theta)\cdot\vec{J^+}} \text{ and } D_{(0,\frac{1}{2})} = e^{(\vec{n}y + i\vec{m}\theta)\cdot\vec{J^-}} \tag{20}$$

which act on two component objects called Weyl spinors. These are mathematical operators that were first invented by Pauli and Dirac in order to describe the physical property of spin [4]. The $(\frac{1}{2},0)$ representation is a left handed spinor $\vec{\chi_l}$ corresponding to the angular momentum of the particle being in the opposite direction of the momentum and the $(0,\frac{1}{2})$ representation corresponding to a right handed spinor $\vec{\chi_r}$. We can identify these with spin $\frac{1}{2}$ fermions due to the angular momentum of the representation but also due to the symmetry of these objects under a rotation of 2π. To examine how this works, the transformations $\vec{J^+}$ and $\vec{J^-}$ in 20 must be defined in a more tangible way by defining the matrices that represent rotations and boosts. These matrices must satisfy the commutation relation for the Lorentz generators and it realized that such a set is composed of the Pauli matrices $(\frac{1}{2}\sigma_i)$. Thus, the finite rotational transformation for either spinor is represented as

$$D = e^{\frac{1}{2}i\vec{m}\cdot\vec{\sigma}\theta} \tag{21}$$

and if we consider a rotation of 2π in the z direction on one of the spinors we find

$$D(\pi/2_z)\vec{\chi_{l,r}} = e^{i\cdot\sigma_3\pi}\vec{\chi_{l,r}} = \begin{pmatrix} e^{i\cdot\sigma_3\pi} & o \\ 0 & e^{i\cdot\sigma_3\pi} \end{pmatrix} \vec{\chi_{l,r}} = -\vec{\chi_{l,r}} \tag{22}$$

Thus it is seen that such a rotation causes for the state of the spinor is only invariant for rotations of 4π which is unique from normal vectors which are invariant under

2π rotations [7]. The idea of spinors can be extended to a more general class called Dirac spinors which are formed as the direct sum of the two Weyl spinors and in general transforms under Lorentz transformations as $(1/2, 0) \oplus (0, 1/2)$ which has four components, two from each fundamental spinor. One interpretation of this type of transformation rule for spinors is that all spin 1/2 Dirac particles are a superposition of the two independent Weyl spinor states.

3.3 Extended Representations

In general, the classification of a mathematical object, and certain physical characteristics of the associated physical particle or field, can be classified by how it transforms under a set of transformations. It is thus reasonable that such objects can be found by simply finding different representations of the Lorentz group.

The first of these can be built using the irreducible Weyl spinor representations and combining them as $(1/2, 0) \otimes (0, 1/2)$. To find the equivalent (j^+, j^-) representation, we use the rules of Clebsch-Gordon coefficient combinations to find that this combination gives the representation (1/2,1/2). This representation has four components and when acted on by rotations transforms like $1/2 \times 1/2 = 1 + 0$ which is identical to the properties of four vectors. In physics, an example of a four vector object which transforms under this representation is that of the four dimensional vector potential A^μ in relativistic electricity and magnetism.

The next step in finding alternate representations is to take two (1/2,1/2) representations and combine then, using Clebsch-Gordon combinations, to find the resulting representations as follows.

$$\begin{aligned} (1/2, 1/2) \otimes (1/2, 1/2) &= (1 \oplus 0, 1 \oplus 0) \\ &= (0, 0) \oplus [(1, 0) \oplus (0, 1)] \oplus (1, 1) \end{aligned} \tag{23}$$

It is seen that four representations result from this combination of the two four vector representations. The first of these is the spin zero representation and so represents scalar particles in field theory. The set of two grouped representations (1,0) and (0,1), transforms as spin 1 objects under rotations. Physically, this can be identified with the electric and magnetic fields if the objects being transformed are the combinations of fields

$$\Sigma^i_{(1,0)} = E^i + iB^i \text{ and } \Sigma^i_{(0,1)} = E^i - iB^i \tag{24}$$

This identification of a spin 1 representation of the Lorentz group with electric and magnetic fields shows that this is the category in which photons live [6]. Lastly, the (1,1) representation is realized to be a spin 2 object and so does not fit into the classes thus defined but should be identified with a symmetric tensor. And, by examining the properties of this tensor object, it can be shown that it obeys Einstein's equation of general relativity

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{25}$$

where $G_{\mu\nu}$ is defined in terms of deviations $h_{\mu\nu}$ which are deviations in space-time from the flat metric. These deviations are the symmetric tensors that transform as the (1,1) representation of the Lorentz group and they are the as yet unobserved graviton particles.

4 Conclusion

Group theory has applications in many areas of physics. In particular, studying Lorentz transformations as a group is very rich in describing classes of particles and gaining insights into fundamental aspects of nature. This has been seen by studying the different classes of objects which emerge through different representations of SO(3,1). However, there are innumerably more paths of study concerning the Lorentz group on which there is well established literature. It goes without saying that much has been discovered since Wigner first realized the physical significance of studying group theory.

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